SOLVING DIFFERENTIAL GEOMETRY

Solutions to Problems Suggested by Dr. Ghomi

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This is a work of math problems. Hints, solutions and answers are the products of the author's efforts. Typos and mistakes are totally unavoidable.

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Preface

The present volume contains hints or full solutions to many of the exercises in two volumes of *Lecture Notes on Differential Geometry* [Gho07] by Mohammad Ghomi, *Professor of Mathematics*, *Georgia Institute of Technology*, and in the first eight chapters of the text *Riemannian Geometry* [Car92] by Manfredo do Carmo, *Emeritus Researcher at the IMPA*. Solving problems being an essential part of the learning process, my goal is to provide those learning and teaching differential geometry with a large number of worked out exercises. Ghomi's notes and do Carmo's textbook cover most "superb" topics in "classical" differential geometry that are usually taught at the undergraduate level: the four vertex theorem, Fairy-Milnor theorem, Gauss's Theorema Egregium, and the Gauss-Bonnet theorem; and topics in "modern" differential geometry that are usually taught at the graduate level: geodesics, the Riemann curvature tensor, Jacobi fields, Hopf-Rinow and Hadamard theorems, and spaces of constant curvature. Therefore this solutions manual can be helpful to anyone learning or teaching differential geometry at both levels.

As Ghomi's notes and do Carmo's textbook contain exercises to test the student's understanding and extend knowledge and insight into the subject, I encourage the reader to work through all of the exercises. To make the solutions concise, I have included only the necessary arguments; the reader may have to fill in the details to get complete proofs.

Comments and questions on possibly erroneous solutions, as well as suggestions for more elegant or more complete solutions will be greatly appreciated.

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Georgia Tech, 2019

Part 1

Curves and Surfaces

Exercise 1 (The simplest proof of the Cauchy-Schwartz inequality). For all p and q in $\mathbf{R}^n$. Prove that the equality holds if and only if $p = \lambda q$ for some $\lambda \in \mathbf{R}$ without using either the quadratic formula or the Lagrange multipliers.

Solution. Assume, as in the second proof above, that $\|p\| = \|q\| = 1$. Then

$$\begin{aligned}
& 0 \leqslant \|p - \langle p, q\rangle\, q\|^2 \\
\iff \quad & 0 \leqslant \langle p - \langle p, q\rangle\, q, p - \langle p, q\rangle\, q\rangle \\
\iff \quad & 0 \leqslant \langle p, p\rangle - (\langle p, q\rangle)^2 - (\langle p, q\rangle)^2 + (\langle p, q\rangle)^2 \|q\|^2 \\
\iff \quad & 0 \leqslant \|p\|^2 - 2(\langle p, q\rangle)^2 + (\langle p, q\rangle)^2 \|q\|^2 \\
\iff \quad & 0 \leqslant 1 - 2(\langle p, q\rangle)^2 + (\langle p, q\rangle)^2 \\
\iff \quad & (\langle p, q\rangle)^2 \leqslant 1,
\end{aligned}$$

giving that

$$\begin{aligned}
|\langle p, q\rangle| &\leq 1 \\
&= \|p\|\,\|q\|\,.
\end{aligned}$$

Let

$$p' = \frac{p}{\|p\|}, \qquad q' = \frac{q}{\|q\|}.$$

Then

$$\|p\|' = \|q\|' = 1.$$

Apply the result above to obtain

$$\left|\left\langle \frac{p}{\|p\|}, \frac{q}{\|q\|}\right\rangle\right| \leqslant 1,$$

so that

$$|\langle p, q\rangle| \leqslant \|p\|\,\|q\|\,.$$

The case $p = 0$ or $q = 0$ is trivial.

Exercise 2 (The triangle inequality). Show that

$$\operatorname{dist}(p, q) + \operatorname{dist}(q, r) \geqslant \operatorname{dist}(p, r)$$

for all p, q in $\mathbf{R}^n$.

SOLUTION (1). Let

$$\begin{aligned} p &= (p_1, p_2, \ldots, p_n), \\ q &= (q_1, q_2, \ldots, q_n), \\ r &= (r_1, r_2, \ldots, r_n). \end{aligned}$$

Then

$$\begin{aligned} &\|p-r\|^2 \\ &= \sum_{i=1}^{n}[(p_i-q_i)+(q_i-p_i)]^2 \\ &= \sum_{i=1}^{n}(p_i-q_i)^2+\sum_{i=1}^{n}(q_i-p_i)^2+2\sum_{i=1}^{n}(p_i-q_i)(q_i-r_i) \\ &= \|p-q\|^2+\|q-r\|^2+2\langle p-q,q-r\rangle \\ &\leqslant \|p-q\|^2+\|q-r\|^2+2\,|\langle p-q,q-r\rangle| \\ &= \|p-q\|^2+\|q-r\|^2+2\,\|p-q\|\,\|q-r\| && \text{by Cauchy-Schwartz inequality} \\ &= (\|p-q\|+\|q-r\|)^2. \end{aligned}$$

Therefore

$$\|p-r\| \leqslant \|p-q\|+\|q-r\|,$$

so that

$$\operatorname{dist}(p;r) \leqslant \operatorname{dist}(p;q)+\operatorname{dist}(q;r),$$

as required.

SOLUTION (2). We have that

$$\begin{aligned} &\|p-r\|^2 \\ &= \langle p-r,p-r\rangle \\ &= \langle (p-q)+(q-r),(p-q)+(q-r)\rangle \\ &= \langle p-q,p-q\rangle+2\langle p-q,q-r\rangle+\langle q-r,q-r\rangle \\ &\leqslant \|p-q\|^2+2\,|\langle p-q,q-r\rangle|+\|q-r\|^2 \\ &\leqslant \|p-q\|^2+2\,\|p-q\|\,\|q-r\|+\|q-r\|^2 && \text{by Cauchy-Schwartz inequality} \\ &= (\|p-q\|+\|q-r\|)^2 \end{aligned}$$

Therefore

$$\|p-r\| \leqslant \|p-q\|+\|q-r\|,$$

so that

$$\operatorname{dist}(p;r) \leqslant \operatorname{dist}(p;q) + \operatorname{dist}(q;r),$$

as required.

Exercise 3. Suppose that p, o, q lie on a line and o lies between p and q. Show that then $\angle poq = \pi$.

Solution. o lies between p and q so there exists the real number $t < 0$ such that $p - o = t(q - o)$. Then

$$\begin{aligned}
\angle poq &:= \cos^{-1} \frac{\langle p - o, q - o\rangle}{\|p - o\| \, \|q - 0\|} \\
&= \cos^{-1} \frac{\langle t(q - o), q - o\rangle}{\|t(q - o)\| \, \|q - 0\|} \\
&= \cos^{-1} \frac{t \, \|q - 0\|^2}{|t| \, \|q - 0\|^2} \\
&= \cos^{-1}(-1) && \text{since } t < 0 \\
&= \pi
\end{aligned}$$

Exercise 4 (Sum of the angles in a triangle). Show that the sum of the angles in a triangle is π (Hint: through one of the vertices draw a line parallel to the opposite side).

Solution. Let have the triangle in the figure. Since lines a and b are parallel, it follows that $\angle BAC = \angle B'CA$ and $\angle ABC = \angle BCA'$. Clearly, $\angle B'CA + \angle ACB + \angle BCA' = 180°$ since $\angle B'CA, \angle ACB, \angle BCA'$ does a complete rotation from one side of the straight line to the other. Hence $\angle ABC + \angle BCA + \angle CAB = 180°$, giving that the sum of the angles in the triangle ABC is $180°$ or π.

Exercise 5. Find a formula for the curve which is traced by the motion of a fixed point on a wheel of radius r rolling with constant speed on a flat surface (Hint: Add the formula for a circle to the formula for a line generated by the motion of the center of the wheel. You only need to make sure that the speed of the line correctly matches the speed of the circle)

Solution (Adapted from George F. Simmons, *Calculus Gems: Brief Lives and Memorable Mathematics*, McGraw-Hill, 1992.). We assume that the rolling circle of radius r rolls along the x-axis, starting from a position in which the center of circle is on the positive y-axis. The curve which is

traced by the rolling circle is the locus of the point P on the circle which was located at the origin O when the center C is on the y-axis. Let θ be the angle through which the radius CP turns as the circle rolls to a new position. If x and y are the coordinates of P, then the rolling of the circle implies that $OB = \text{arclength}\, BP = r\theta$, so $x = OB - AB = OB - PQ = r\theta - r\sin\theta = r(\theta - \sin\theta)$. Also, $y = BC - QC = r - r\cos\theta = r(1-\cos\theta)$. The desired curve therefore has the reparametrization

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} \alpha(t) = (r(t - \sin t), r(1 - \cos t)).$$

This is a cycloid.

EXERCISE 6. Show that if the position vector and velocity of a planar curve $\alpha \colon I \to \mathbf{R}^2$ are always perpendicular, i.e., $\langle \alpha(t), \alpha'(t)\rangle$, for all $t \in I$, then $\alpha(I)$ lies on a circle centered at the origin of $\mathbf{R}^2$.

SOLUTION. Let $(x(t), y(t))$ be the position vector, then the velocity vector $\alpha'(t) = (x'(t), y'(t))$.

$$\begin{aligned} & & \big\langle \alpha(t), \alpha'(t)\big\rangle &= 0 \\ &\Longleftrightarrow & x(t)x'(t) + y(t)y'(t) &= 0 \\ &\Longleftrightarrow & \frac{d}{dt}(x^2(t) + y^2(t)) &= 0 \\ &\Longleftrightarrow & x^2(t) + y^2(t) &= C, \end{aligned}$$

for some constant C. Thus $\alpha(I)$ lies on a circle centered at the origin of $\mathbf{R}^2$.

EXERCISE 7. Compute the length of a circle of radius r, and the length of one cycle of the curve traced by a point on a circle of radius r rolling on a straight line.

SOLUTION. The reparametrization of a circle centered at the origin having radius r is

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} \alpha(t) = (r\cos t, r\sin t).$$

Then

$$\alpha'(t) = (-r\sin t, r\cos t).$$

The length of this circle is

$$\begin{aligned}
\text{length} &= \int_0^{2\pi} \left\| \alpha'(t) \right\| \, dt \\
&= \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} \, dt \\
&= r \int_0^{2\pi} \, dt && \text{since } \sin^2 t + \cos^2 t = 1 \\
&= 2\pi r.
\end{aligned}$$

The reparametrization of the first cycle of a cycloid through the origin, with a horizontal base given by the line $y = 0$, is

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} \alpha(t) = (r(t - \sin t), r(1 - \cos t)).$$

Then

$$\alpha'(t) = (r - r\cos t, r\sin t).$$

The length of one cycle of this cycloid is

$$
\begin{aligned}
\text{length} &= \int_0^{2\pi} \left\|\alpha'(t)\right\| \, dt \\
&== \int_0^{2\pi} \sqrt{(r - r\cos t)^2 + (r\sin t)^2} \, dt \\
&= r\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t} \, dt \\
&= r\int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \, dt. \\
&= r\int_0^{2\pi} \sqrt{2(1-\cos t)} \, dt \\
&= r\int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}} \, dt && \text{since } 1 - \cos t = 2\sin^2 \frac{t}{2} \\
&= r\int_0^{2\pi} 2\sin \frac{t}{2} \, dt \\
&= -4r\cos \frac{t}{2}\Big|_0^{2\pi} \\
&= 4r - (-4r) \\
&= 8r.
\end{aligned}
$$

EXERCISE 8. Show that if $\alpha\colon [a,b] \to \mathbf{R}^2$ is a closed curve with width w and length L, then

$$
w \leq \frac{L}{\pi}.
$$

SOLUTION. We prove this by contradiction. Assume that $L < \pi w$. Let $\mathscr{H}$ be the convex hull of the set of points bounded by the closed curve α. Let length$[MN]$ be the width of $\mathscr{H}$, where M and N are points in the boundary β of $\mathscr{H}$. Stretch β into a circle C_α. Then length$[C_\alpha] \leq L$ since concave arcs are replaced by straight segments. The radius of C_α is

$$
r_\alpha = \frac{\text{length}[C_\alpha]}{2\pi} \leq \frac{L}{2\pi} < \frac{\pi w}{2\pi} = \frac{w}{2}.
$$

This implies that length$[MN] \leq 2r_\alpha < w$, contradicting the the minimality of w as the width of α. Therefore

$$w \leq \frac{L}{\pi}.$$

EXERCISE 9. Show that if the equality in the exercise above holds then is a curve of constant width.

SOLUTION. Suppose the equality in the exercise above holds, that is,

$$w = \frac{L}{\pi},$$

which is a constant. Therefore the curve has a constant width, for example, the circle of radius $w/2$. The reverse is also true. It has been known since Barbier in 1860[1], and follows from the Cauchy-Crofton formula, that for closed planar curves $L/w \geq \pi$, where equality holds only for curves of constant width. (see Ghomi, Mohammad. (2016). The length, width, and inradius of space curves. Geometriae Dedicata. 10.1007/s10711-017-0312-3).

EXERCISE 10. Show that the curvature of a circle of radius r is $\frac{1}{r}$, and the curvature of the line is zero (First you need to find arclength parametrizations for these curves).

SOLUTION. We saw earlier in Section 1.1 of Lecture Notes 1 that the parametrization of a circle of radius r with respect to arc length was

$$[0, 2\pi] \ni t \xmapsto{\alpha} (r\cos(t), r\sin(t)) \in \mathbf{R}^2.$$

First, we need to compute $T(t)$. By definition,

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

So, we must first compute $\alpha'(t)$.

$$\begin{aligned} \alpha'(t) &= (r(-\frac{1}{r}\sin(\frac{t}{r})), r(\frac{1}{r}\cos(\frac{t}{r}))) \\ &= (-\sin(\frac{t}{r}), \cos(\frac{t}{r})). \end{aligned}$$

[1]E. Barbier. Note sur le probl'eme de l'aiguille et le jeu du joint couvert. *Journal de math ́ematiques pures et appliqu ́ees*, pages 273–286, 1860.

We can see that

$$\begin{aligned}\left\|\alpha'(t)\right\| &= \sqrt{[-\sin(\frac{t}{r})]^2 + [\cos(\frac{t}{r})]^2} \\ &= \sqrt{\sin^2(\frac{t}{r}) + \cos^2(\frac{t}{r})} = 1.\end{aligned}$$

Thus

$$T(t) = (-\sin(\frac{t}{r}), \cos(\frac{t}{r})).$$

It follows that

$$T'(t) = (-\frac{1}{r}\cos(\frac{t}{r}), -\frac{1}{r}\sin(\frac{t}{r})),$$

and therefore

$$\begin{aligned}\kappa(t) &= \left\|T'(t)\right\| \\ &= \sqrt{(-\frac{1}{r}\cos(\frac{t}{r}))^2 + (-\frac{1}{r}\sin(\frac{t}{r}))^2} \\ &= \sqrt{\frac{1}{r^2}(\cos^2(\frac{t}{r}) + \sin^2(\frac{t}{r}))} \\ &= \frac{1}{r}.\end{aligned}$$

In other words, the curvature of a circle is the inverse of its radius. This agrees with our intuition of curvature. Curvature is supposed to measure how sharply a curve bends. The larger the radius of a circle, the less it will bend, that is the less its curvature should be. This is indeed the case. The larger the radius, the smaller its inverse.

Earlier also in Section 1.1 of Lecture Notes 1, we found that the parametrization of a line which passes through a point $p \in \mathbf{R}^n$and is parallel to the vector $v \in \mathbf{R}^n$, with respect to arc length was

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} p + tv \in \mathbf{R}^n.$$

As before, we need to compute $T(t)$ which can be obtained from $\alpha'(t)$.

$$\alpha'(t) = v \in \mathbf{R}^n.$$

We can see that

$$\left\|\alpha'(t)\right\| = \|v\| \in \mathbf{R}.$$

Thus

$$T(t) = \frac{v}{\|v\|} \in \mathbf{R}^n.$$

Notice that each component of $T(t)$ is a scalar, so differentiating $T(t)$, we get

$$T'(t) = (0, 0, \ldots, 0) \in \mathbf{R}^n,$$

and therefore

$$\begin{aligned} \kappa(t) &= \left\|T'(t)\right\| \\ &= 0. \end{aligned}$$

EXERCISE 11. Show that the curvature of a planar curve which satisfies the equation $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{(\sqrt{1 + (f'(x))^2})^3}.$$

(*Hint*: Use the parametrization $\alpha(t) = (t, f(t), 0)$, and use the formula in previous exercise.) Compute the curvatures of $y = x$, x^2, x^3, and x^4.

SOLUTION. First, let us remark that it is easy to parametrize the curve given by $y = f(x)$ as a parametrized curve. We can simply use

$$\begin{cases} x = x \\ y = f(x) \\ z = 0. \end{cases}$$

Using t as the name if the parameter. Thus, the position vector of our curve $\alpha(t) = (t, f(t), 0)$. It follows that

$$\alpha'(t) = (1, f'(x), 0),$$

and

$$\alpha''(t) = (0, f''(x), 0).$$

Thus

$$\alpha'(t) \times \alpha''(t) = (0, 0, f''(x))$$

Hence

$$\left\|\alpha'(t) \times \alpha''(t)\right\| = \left\|f''(x)\right\|$$

and

$$\left\|\alpha'(t)\right\| = \sqrt{1 + (f'(x))^2}.$$

Therefore, by the formula

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$$

in Exercise 9 of Lecture Notes 2, we obtain

$$\kappa(x) = \frac{|f''(x)|}{(\sqrt{1 + (f'(x))^2})^3}.$$

We illustrate these formulas with some examples. If $y = x$, then

$$\begin{aligned}
\kappa(x) &= \frac{|f''(x)|}{(\sqrt{1 + (f'(x))^2})^3} \\
&= \frac{|0|}{(\sqrt{1 + 1^2})^3} \\
&= 0.
\end{aligned}$$

If $y = x^2$, then

$$
\begin{aligned}
\kappa(x) &= \frac{|f''(x)|}{(\sqrt{1+(f'(x))^2})^3} \\
&= \frac{|2|}{(\sqrt{1+(2x)^2})^3} \\
&= \frac{2}{(\sqrt{1+4x^2})^3}.
\end{aligned}
$$

If $y = x^3$, then

$$
\begin{aligned}
\kappa(x) &= \frac{|f''(x)|}{(\sqrt{1+(f'(x))^2})^3} \\
&= \frac{|6x|}{(\sqrt{1+(3x^2)^2})^3} \\
&= \frac{|6x|}{(\sqrt{1+9x^4})^3}.
\end{aligned}
$$

If $y = x^4$, then

$$
\begin{aligned}
\kappa(x) &= \frac{|f''(x)|}{(\sqrt{1+(f'(x))^2})^3} \\
&= \frac{\left|12x^2\right|}{(\sqrt{1+(4x^3)^2})^3} \\
&= \frac{12x^2}{(\sqrt{1+16x^6})^3}.
\end{aligned}
$$

EXERCISE 12. Let $\alpha, \beta \colon (-1,1) \to \mathbb{R}^2$ be a pair of C^2 curves with $\alpha(0) = \beta(0) = (0,0)$. Further suppose that α and β both lie on or above the x-axis, and β lies higher than or at the same height as α. Show that the curvature of β at $t = 0$ is not smaller than that of α at $t = 0$.

SOLUTION. Let $v(t)$ be the tangent at $t = 0$ of α. Consider the rotation

$$
R_1 = \begin{pmatrix} \cos(-\tan^{-1} v(0)) & -\sin(-\tan^{-1} v(0)) \\ \sin(-\tan^{-1} v(0)) & \cos(-\tan^{-1} v(0)) \end{pmatrix}.
$$

Then $R_1 \circ \alpha$ has the tangent at $t = 0$ be the x-axis. Hence

$$(R_1 \circ \alpha)'(0) = 0.$$

Let $\kappa_\alpha(t)$ be the curvature of α. Then

$$K_\alpha(0)\,|(R_1 \circ \alpha)(0)| \qquad \text{by the formula } \kappa(t) = \frac{|f''(t)|}{(\sqrt{1 + (f'(x))^2})^3}.$$

Similarly, let R_2 be the rotation for β, then since $\beta \geq \alpha$, so consider an open neighborhood V sufficiently small and use Taylor expansion to obtain

$$0 \leq (R_2 \circ \beta - R_1 \circ \alpha)(t) = \frac{1}{2}[(R_2 \circ \beta - R_1 \circ \alpha)(0)]''t^2 + o(t^2) \quad \text{since } \beta \geq \alpha,$$

implying

$$(R_2 \circ \beta)''(0) \geq (R_1 \circ \alpha)''(0) \geq 0$$

since $R_1 \circ \alpha$ lies above the x-axis, so it is convex in some neighborhood U of 0, so $(R_1 \circ \alpha)'' \geq 0$. Moreover,

$$\begin{aligned} \kappa_\beta(0) &= \left|(R_2 \circ \beta)''(0)\right| \\ &= (R_2 \circ \beta)''(0) \\ &\geq (R_1 \circ \alpha)''(0) \\ &= \left|(R_1 \circ \alpha)''(0)\right| \\ &= \kappa_\alpha(0). \end{aligned}$$

Therefore $\kappa_\beta(0) \geq \kappa_\alpha(0)$.

Exercise 13. Show that if $\alpha\colon I \to \mathbb{R}^2$ is a C^2 closed curve which is contained in a circle of radius r, then the curvature of α has to be bigger than $1/r$ at some point. In particular, closed curves have a point of nonzero curvature. (*Hint*: Shrink the circle until it contacts the curve, and use the exercise above).

Solution. Consider an arbitrary point$\alpha(t_0)$ on the closed curve α. Shrink the circle C until it contacts the curve α at the point $\alpha(t_0)$. Let C_1 be the shrunk circle. Then the radius r_1 of the circle C_1 is smaller or equal to the radius r of the circle C. Define the rectangular coordinate system that originate at the point $\alpha(t_0)$. Exercise 11 of Lecture Note 1 gives that $\kappa(t_0) \geq \frac{1}{r_1} \geq \frac{1}{r}$ where $\kappa(t)$ is the curvature of the closed curve α.

EXERCISE 14. Let $\alpha\colon I \to \mathbf{R}^2$ be a closed planar curve, show that

$$\operatorname{length}[\alpha] \geq \frac{2\pi}{\max \kappa}$$

(*Hint*: Recall that the width w of α is smaller than or equal to its length divided by π to show that a piece of α should lie inside a circle of diameter at least w).

SOLUTION. Recall that the width w of α is smaller than or equal to its length L divided by π, that is,

$$w \leq \frac{L}{\pi},$$

implying

$$L \geq \pi w. \tag{1}$$

The definition of the width of a curve gives that a piece of α should lie inside a circle of diameter at least w. For any point $\alpha(t)$ on this piece, the exercise above gives the following inequality

$$\kappa(t) \geq \frac{1}{w/2} = \frac{2}{w},$$

implying

$$w \geq \frac{2}{\kappa(t)} \geq \frac{2}{\max \kappa(t)}. \tag{2}$$

It follows from (1) and (2) that

$$L \geq \frac{2\pi}{\max \kappa(t)}.$$

EXERCISE 15. Show that the only curves with constant zero curvature in $\mathbf{R}^n$ are straight lines. (*Hint*: We may assume that our curve, $\alpha\colon I \to \mathbf{R}^n$ has unit speed. Then $\kappa = \|\alpha''\|$. So zero curvature implies that $\alpha = 0$. Integrating the last expression twice yields the desired result.)

SOLUTION. We may assume that our curve, $\alpha\colon I \to \mathbf{R}^n$ has unit speed. If the curve α is parametrized with respect to arc length then by definition $\kappa = \|\alpha''\| = 0$. This is equivalent to $\alpha'' = 0$. Integrating the last expression

to obtain $\alpha' = v \in \mathbf{R}^n$ where $\|v\| = 1$. Integrating the last expression to obtain

$$\alpha(t) = p + vt,$$

where $\|v\| = 1$, $p \in \mathbf{R}^n$. Thus the curve α is a straight line.

Exercise 16. Show that $T(t)$ and $N(t)$ are orthogonal. (*Hint*: Differentiate both sides of the expression $\langle T(t), T(t)\rangle = 1$).

Solution. We have

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

Thus

$$\begin{aligned} \|T(t)\| &= \frac{1}{\|\alpha'(t)\|} \cdot \left\|\alpha'(t)\right\| \\ &= 1. \end{aligned}$$

Hence

$$\begin{aligned} \langle T(t), T(t)\rangle &= \|T(t)\|^2 \\ &= 1. \end{aligned}$$

Differentiate both sides of the last expression to obtain

$$2\left\langle T(t), T'(t)\right\rangle = 0$$

since

$$\left\langle T'(t), T'(t)\right\rangle = \left\langle T(t), T'(t)\right\rangle$$

and

$$(\langle u(t), v(t)\rangle)' = \left\langle u'(t), v(t)\right\rangle + \left\langle u(t), v'(t)\right\rangle.$$

This implies that

$$\left\langle T(t), \left\|T'(t)\right\| N(t)\right\rangle = 0$$

since $N(t) = \frac{T'(t)}{\|T'(t)\|}$. In other words,

$$\left\|T'(t)\right\| \langle T(t), N(t)\rangle = 0.$$

Thus $\langle T(t), N(t)\rangle = 0$, that is, $T(t)$ and $N(t)$ are orthogonal.

EXERCISE 17. Check that the osculating circle of α is tangent to α at $\alpha(t)$ and has the same curvature as α at time t.

SOLUTION. The osculating circle equation of α is

$$\left\|s(u) - \alpha(t) - \frac{1}{\kappa(t)}N(t)\right\| = \frac{1}{\kappa(t)},$$

where $u \in [0, 2\pi]$. The point $\alpha(t)$ satifies this equation, so $\alpha(t) \in s(u)$ (note that $\|N(t)\| = 1$, so $\left\|\frac{1}{\kappa(t)}N(t)\right\| = \frac{1}{\kappa(t)}$). From the osculating circle equation, we have

$$\left\langle s(u) - \alpha(t) - \frac{1}{\kappa(t)}N(t), s(u) - \alpha(t) - \frac{1}{\kappa(t)}N(t)\right\rangle = \frac{1}{\kappa^2(t)}.$$

Differentiate both sides of the last expression with respect to u to obtain

$$\left\langle s'(u), s(u) - \alpha(t) - \frac{1}{\kappa(t)}N(t)\right\rangle = 0,$$

for all u. Since $\alpha(t) \in s(u)$, there exists $u_0 \in [0, 2\pi]$ such that $s(u_0) = \alpha(t)$ and the expression above at time u_0 is

$$\left\langle s'(u_0), s(u) - \alpha(t) - \frac{1}{\kappa(t)}N(t)\right\rangle = 0$$

or

$$\left\langle s'(u_0), -\frac{1}{\kappa(t)}N(t)\right\rangle = 0.$$

This implies

$$\frac{1}{\kappa(t)}\left\langle s'(u_0), N(t)\right\rangle = 0,$$

so that

$$\big\langle s'(u_0), N(t) \big\rangle = 0.$$

Hence the tangent of $s(u)$ at the point $\alpha(t)$ coincides with $T(t)$, so $s(u)$ is tangent to α at the point $\alpha(t)$.

The curvature of the osculating circle of α is

$$\overline{\kappa}(t) = \frac{1}{1/\kappa(t)} = \kappa(t)$$

(since the osculating circle of α has radius $1/\kappa(t)$). Thus the osculating circle of α and α have the same curvature at time t.

EXERCISE 18. Show that if α has constant curvature c, then (i) $p(t)$ is a fixed point, and (ii) $\alpha(t) - p(t) = 1/c$ (*Hint*: For part (i) differentiate $p(t)$; part (ii) follows immediately from the definition of $p(t)$).

SOLUTION. (i) Suppose that α has constant curvature c. We can assume further that the curve α is parametrized with respect to arc length then by definition

$$p(t) = \alpha(t) + \frac{1}{c}N(t).$$

Differentiate both sides of the last expression to obtain

$$\begin{aligned}
p'(t) &= \alpha'(t) + \frac{1}{c}N'(t). \\
&= T(t) + \frac{1}{c}N'(t) && \text{since } T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|} = \alpha'(t) \\
&= T(t) + \frac{1}{c}(-cT(t)) && \text{since } N'(t) = -cT(t) \\
&= 0
\end{aligned}$$

for all t. Hence $p(t)$ is a fixed point.

(ii) From the definition of $p(t)$, we have

$$\begin{aligned}\|\alpha(t)-p(t)\| &= \left\|-\frac{1}{c}N(t)\right\| \\ &= \frac{1}{c}\|N(t)\| \\ &= \frac{1}{c}\end{aligned}$$

(since $\|N(t)\| = 1$). Thus

$$\|\alpha(t)-p(t)\| = \frac{1}{c}.$$

EXERCISE 19. Show that

$$\overline{\kappa}(t) := \frac{\langle \gamma'(t)\times\gamma''(t),(0,0,1)\rangle}{\|\gamma'(t)\|^3}.$$

SOLUTION. Since $\gamma(t)$ is a planar curve, it may be seen as a curve in $\mathbf{R}^3$ as $\gamma(t) = (x(t), y(t), 0) \in \mathbf{R}^3$. Then

$$\gamma'(t) = (x'(t), y'(t), 0).$$

Thus

$$\begin{aligned}T(t) &= \frac{\gamma'(t)}{\|\gamma'(t)\|} \\ &= \frac{(x'(t), y'(t), 0)}{\sqrt{(x'(t))^2+(y'(t))^2}}.\end{aligned}$$

Hence

$$T'(t) = \frac{1}{\|\gamma'(t)\|^2}(x''(t)\,\|\gamma'(t)\| - \frac{x'(t)[x'(t)+y'(t)]}{\|\gamma'(t)\|}, y''(t)\,\|\gamma'(t)\| - \frac{y'(t)[x'(t)+y'(t)]}{\|\gamma'(t)\|}, 0).$$

So

$$iT(x) = \frac{1}{\|\gamma'(t)\|}(-y'(t), x'(t), 0).$$

Therefore

$$\begin{aligned}
\overline{\kappa}(t) &= \frac{\langle T'(t), iT \rangle}{\|\gamma(t)\|} \\
&= \frac{1}{\|\gamma'(t)\|^4}(-x''(t)y'(t)\,\|\gamma'(t)\| + x'(t)y''(t)\,\|\gamma'(t)\|) \\
&= \frac{x'y'' - x''y'}{\|\gamma'(t)\|^3}. \qquad (1)
\end{aligned}$$

Note that

$$\gamma'(t) = (x'(t), y'(t), 0),$$

giving

$$\gamma''(t) = (x''(t), y''(t), 0).$$

Thus

$$\begin{aligned}
\gamma'(t) \times \gamma''(t) &= (\begin{vmatrix} y'(t) & 0 \\ y''(t) & 0 \end{vmatrix}, \begin{vmatrix} 0 & x'(t) \\ 0 & x''(t) \end{vmatrix}, \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{vmatrix}) \\
&= (0, 0, x'y'' - x''y').
\end{aligned}$$

Hence

$$\frac{\langle \gamma'(t), \gamma''(t), (0,0,1) \rangle}{\|\gamma'(t)\|^3} = \frac{x'y'' - x''y'}{\|\gamma'(t)\|^3}. \qquad (2)$$

Combine (1) and (2) to obtain

$$\overline{\kappa}(t) := \frac{\langle \gamma'(t) \times \gamma''(t), (0,0,1) \rangle}{\|\gamma'(t)\|^3}.$$

Exercise 20. (i) Compute the total curvature and rotation index of a circle which has been oriented clockwise, and a circle which is oriented counterclockwise. Sketch the figure eight curve $(\cos t, \sin 2t)$, $0 \le t \le 2\pi$, and compute its total signed curvature and rotation index.

Solution. The parametrization of a circle, which has been oriented clockwise, centered at the origin having radius r is

$$[0, 2\pi] \ni t \xmapsto{\alpha} \alpha(t) = (r\cos(-t), r\sin(-t)).$$

The arclength from time 0 to time t is

$$\begin{aligned}
s(t) &= \int_0^t \left\| \alpha'(u) \right\| \, du \\
&= \int_0^t \|(r\sin(-t), -r\cos(-t)\| \, du \\
&= \int_0^t \sqrt{r^2[\sin^2(-t) + \cos^2(-t)]} \, du \\
&= \int_0^t r \, du \\
&= rt,
\end{aligned}$$

giving

$$t = \frac{s(t)}{r}.$$

The reparametrization by arclength of a circle, which has been oriented clockwise, centered at the origin having radius r is

$$[0, 2\pi r] \ni s \xmapsto{\beta} \beta(s) = (r\cos(-\frac{s}{r}), r\sin(-\frac{s}{r})) = \beta(s) = (r\cos(\frac{s}{r}), -r\sin(\frac{s}{r})),$$

so

$$\begin{aligned}
\beta'(s) &= (-\sin(\frac{s}{r}), -\cos(\frac{s}{r})) \\
\beta''(s) &= (-\frac{1}{r}\cos(\frac{s}{r}), \frac{1}{r}\sin(\frac{s}{r})).
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa(s) &= \left\| \beta''(s) \right\| \\
&= \sqrt{\frac{1}{r^2}[\cos^2(\frac{s}{r}) + \sin^2(\frac{s}{r})]} \\
&= \frac{1}{r}.
\end{aligned}$$

The total curvature of a clockwise circle is

$$\begin{aligned}
\text{total}\kappa[\alpha] &= \int_0^{2\pi r} \kappa(s)\, ds \\
&= \int_0^{2\pi r} \frac{1}{r}\, ds \\
&= \frac{2\pi r}{r} = 2\pi.
\end{aligned}$$

Note that the curvatures of circles of the same radius are the same, so the total curvature of a clockwise circle equals to the total curvature of a counterclockwise circle, that is, 2π.

The parametrization of a circle, which has been oriented counterclockwise, centered at the origin having radius r is

$$[0, 2\pi] \ni t \stackrel{\alpha}{\longmapsto} \alpha(t) = (r\cos t, r\sin t),$$

so

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t) \\
\alpha''(t) &= (-r\cos t, -r\sin t).
\end{aligned}$$

The signed curvature of α at time t is

$$\begin{aligned}
\overline{\kappa}(t) &= \frac{x'y'' - x''y'}{\|\gamma'(t)\|^3} \\
&= \frac{r^2\sin^2 t + r^2\cos^2 t}{(\sqrt{r^2(\sin^2 t + \cos^2 t})^3} \\
&= \frac{1}{r}.
\end{aligned}$$

The total signed curvature of a counterclockwise circle is

$$\begin{aligned}
\text{total}\kappa[\alpha] &= \int_0^{2\pi r} \overline{\kappa}(s)\, ds \\
&= \int_0^{2\pi r} \frac{1}{r}\, ds \\
&= \frac{2\pi r}{r} = 2\pi.
\end{aligned}$$

The rotation index of a counterclockwise circle is

$$\text{rot}[\alpha] = \frac{\text{total}\kappa[\alpha]}{2\pi} = \frac{2\pi}{2\pi} = 1.$$

The parametrization of a circle, which has been oriented clockwise, centered at the origin having radius r is

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} \alpha(t) = (r\cos t, -r\sin t),$$

so

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, -r\cos t) \\
\alpha''(t) &= (-r\cos t, r\sin t).
\end{aligned}$$

The signed curvature of α at time t is

$$\begin{aligned}
\overline{\kappa}(t) &= \frac{x'y'' - x''y'}{\|\gamma'(t)\|^3} \\
&= \frac{-r^2\sin^2 t - r^2\cos^2 t}{(\sqrt{r^2(\sin^2 t + \cos^2 t})^3} \\
&= -\frac{1}{r}.
\end{aligned}$$

The total signed curvature of a clockwise circle is

$$\begin{aligned}
\text{total}\kappa[\alpha] &= \int_0^{2\pi r} \overline{\kappa}(s)\, ds \\
&= \int_0^{2\pi r} -\frac{1}{r}\, ds \\
&= -\frac{2\pi r}{r} = -2\pi.
\end{aligned}$$

The rotation index of a clockwise circle is

$$\text{rot}[\alpha] = \frac{\text{total}\kappa[\alpha]}{2\pi} = \frac{-2\pi}{2\pi} = -1.$$

We spit the figure eight circle into two parts. Then, by Theorem 13 (Hopf) of Lecture Note 4, the rotation index of the figure eight curve is $+2$, giving the total signed curvature of the figure eight curve is $2 \cdot (2\pi) = 4\pi$.

Exercise 21. Use the above formula to show that the only closed curves of constant curvature in the plane are circles.

Solution. If the closed curve has constant curvature, then $\overline{\kappa}(t) = c$ where c is some constant and $c \neq 0$. This implies that $\theta'(t) = c$, so

$$\begin{aligned}
\theta(t) &= \int_0^t c\, ds + \theta(0) \\
&= ct + \theta(0).
\end{aligned}$$

We have that

$$\begin{aligned}
\alpha(t) &= (\int_0^t \cos\theta(s)\, ds, \int_0^t \sin\theta(s)\, ds) + \alpha(0) \\
&= (\int_0^t \cos(\theta_0 + cs)\, ds, \int_0^t \sin(\theta_0 + cs)\, ds) + \alpha(0) && \text{where } \theta_0 = \theta(0) \\
&= (\frac{1}{c}\sin ct, -\frac{1}{c}\cos ct) + \alpha(0) && \text{choose } \theta_0 = 0 \\
&= (x_0 + \frac{1}{c}\sin ct, y_0 + \frac{1}{c}\cos ct) && \text{where } \alpha(0) = (x_0, y_0).
\end{aligned}$$

Thus $\alpha(t)$ is a circle with the center at (x_0, y_0) and radius $r = 1/|c|$.

The converse is trivial.

EXERCISE 22. Show that if $\alpha\colon I \to \mathbf{R}^2$ is a C^4 curve with monotone nonvanishing curvature, then its evolute is a regular curve which also has nonvanishing curvature. In particular contains no line segments.

SOLUTION. For the first part, we may asume that $\|\alpha'(t)\| = 1$. We have that

$$\beta(t) = \alpha(t) + r(t)N(t).$$

Differentiate both sides of the last expression to obtain

$$\begin{aligned}
\beta'(t) &= \alpha'(t) + r'(t)N(t) + r(t)N'(t) \\
&= \alpha'(t) + r'(t)N(t) - T(t) && \text{since } N'(t) = -\frac{1}{r(t)}T(t) \\
&= \alpha'(t) + r'(t)N(t) - \alpha'(t) && \text{since } T(t) = \alpha'(t) \\
&= r'(t)N(t).
\end{aligned}$$

Thus

$$\begin{aligned}
\big\|\beta'(t)\big\| &= \big\|r'(t)N(t)\big\| \\
&= \big|r'(t)\big|\,\|N(t)\| \\
&= \big|r'(t)\big| && \text{since } \ \|N(t)\| = 1 \\
&= \frac{|\kappa'(t)|}{[\kappa(t)]^2} \\
&\neq 0 && \text{for all } t.
\end{aligned}$$

Hence $\beta'(t) \neq 0$ for all t. Therefore β is a regular curve.

For the last part, suppose, towards a contradiction, that β has vanishing curvature. Then $\beta'(t) = \mathit{const}$, which contradicts the fact that

$$\big\|\beta'(t)\big\| = \frac{|\kappa'(t)|}{[\kappa(t)]^2}$$

is not constant. Thus β has nonvanishing curvature, so it does not contain any line segments.

EXERCISE 23. Show that a curve with monotone curvature cannot have any bitangent lines.

SOLUTION. Suppose, towards a contradiction, that a curve α with monotone curvature has a bitangent line. Then then there are two osulating circles of α have the same radius, which contradicts the fact that osculating circles of α are pairwise disjoint by Kneser's Nesting Theorem.

EXERCISE 24. Show that an ellipse has exactly 4 vertices, unless it is a circle.

SOLUTION. The parametrization of an ellipse having the radius a and b on the x and y axes respectively, is

$$[0,2\pi] \ni t \overset{\alpha}{\longmapsto} \alpha(t) = (a\cos t, b\sin t).$$

The signed curvature of this ellipse is

$$\begin{aligned}
\overline{\kappa}(t) &= \frac{x'(t)y''(t) - x''(t)y'(t)}{[(x'(t))^2 + (y'(t))^2]^{3/2}} \\
&= \frac{-a\sin t(-b\sin t) + a\cos t(b\cos t)}{[(-a\sin t)^2 + (b\cos t)^2]^{3/2}} \\
&= \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} \\
&= \frac{ab}{[a^2(1-\cos^2 t) + b^2\cos^2 t]^{3/2}} \\
&= \frac{ab}{[a^2 + (b^2 - a^2)\cos^2 t]^{3/2}}.
\end{aligned}$$

Thus

$$\min \overline{\kappa}(t) = \begin{cases} \frac{a}{b^2} & \text{if } b > a \qquad (\cos t = \pm 1), \\ \frac{b}{a^2} & \text{if } b \le a \qquad (\cos t = 0), \end{cases}$$

and

$$\max \overline{\kappa}(t) = \begin{cases} \frac{b}{a^2} & \text{if } b > a \qquad (\cos t = 0), \\ \frac{a}{b^2} & \text{if } b \le a \qquad (\cos t \pm 1). \end{cases}$$

Since the signed curvature of the ellipse has local max and local min at 4 points, it follows that the ellipse has 4 vertices.

EXERCISE 25. Verify the following sentence: A simple closed convex curve has at least four vertices.

SOLUTION. This is The Four-Vertex Theorem. The following proof for this theorem is from [Car16].

Before starting the proof, we need a *Lemma*:

Let $\alpha\colon [0,l] \to \mathbf{R}^2$ be a plane closed curve parametrized by arc length and let A, B, and C be arbitrary real numbers. Then

$$\int_0^l (Ax + By + C)\,\frac{d\kappa}{ds}\,ds = 0, \tag{1}$$

where the functions $x = x(s)$, $y = y(s)$ are given by $\alpha(s) = (x(s), y(s))$, and κ is the curvature of α.

Proof of the Lemma. Recall that there exists a differentiable function $\theta\colon [0,l] \to \mathbf{R}$ such that $x\prime(s) = \cos\theta$, $y\prime(s) = \sin\theta$. Thus, $\kappa(s) = \theta'(s)$ and

$$x'' = -ky', \qquad\qquad y'' = kx'.$$

Therefore, since the functions involved agree at 0 and l,

$$\begin{aligned}
&\int_0^l k'\,ds = 0,\\
&\int_0^l xk'\,ds = -\int_0^l kx'\,ds = -\int_0^l y''\,ds = 0,\\
&\int_0^l yk'\,ds = -\int_0^l ky'\,ds = \int_0^l x''\,ds = 0.
\end{aligned}$$

Proof of the Theorem. Parametrize the curve by arc length, $\alpha\colon [0,l] \to \mathbf{R}^2$. *Since $\kappa = \kappa(s)$ is a continuous function on the closed interval $[0,l]$, it reaches a maximum and a minimum on $[0,l]$.* Thus, α has at least two vertices, $\alpha(s_1) = p$ and $\alpha(s_2) = q$. Let L be the straight line passing through p and q, and let β and γ be the two arcs of α which are determined by the points p and q.

We claim that each of these arcs lies on a definite side of L. Otherwise, it meets L in a point r distinct from p and q (Fig. 1(a)). By convexity, and since p, q, r are distinct points on C, the tangent line at the intermediate point, say p, has to agree with L. Again, by convexity, this implies that L is tangent to C at the three points p, q, and r. But then the tangent to a point near p (the intermediate point) will have q and r on distinct sides,

unless the whole segment rq of L belongs to C (Fig. 1(b)). This implies that $\kappa = 0$ at p and q. Since these are points of maximum and minimum for κ, $\kappa \equiv 0$ on C, a contradiction.

Let $Ax + By + C = 0$ be the equation of L. If there are no further vertices, $\kappa'(s)$ keeps a constant sign on each of the arcs β and γ.We can then arrange the signs of all the coefficients A, B, C so that the integral in Eq. (1) is positive. This contradiction shows that there is a third vertex and that $\kappa'(s)$ changes sign on β or γ, say, on β. Since p and q are points of maximum and minimum, $\kappa'(s)$ changes sign twice on β. Thus, there is a fourth vertex.

Exercise 26. Prove the four vertex theorem for convex curves using the Schur's arm lemma.

Solution. Let L be the length of the curve α. Let α_1 be the curve given by

$$\alpha_1 = \alpha|_{[0, \frac{L}{2}]},$$

i.e., the parametrization of α_1 is

$$[0, \frac{L}{2}] \ni t \overset{\alpha_1}{\longmapsto} \alpha_1(t) = \alpha(t),$$

and α_2 be the curve given by

$$\alpha_2 = \alpha|_{[\frac{L}{2}, L]},$$

i.e., the reparametrization of α_2 is

$$[0, \frac{L}{2}] \ni t \overset{\alpha_2}{\longmapsto} \alpha_2(t) = \alpha(\frac{L}{2} + t).$$

Since $\kappa_1 \colon [0, \frac{L}{2}] \to \mathbf{R}$, the curvature of the curve α_1, is continuous and $[0, \frac{L}{2}]$ is a compact set, so $\kappa_1(t)$ attains its maximum and mininum on $[0.\frac{L}{2}]$. Thus α_1 has at least two vertices. Without loss of generality, we can assume that κ_1 attains its maximum and minimum at t_1 and t_2, respectively, such that $0 \le t_1 < t_2 \le \frac{L}{2}$. We can assume that $t_1 = 0$ by choosing the curve α_1 so that α_1 starts at $t_1 = 0$. If α has only two vertices above, then $\kappa_2'(t) > 0$. This implies that $\kappa_2(t) \ge \kappa_2(\frac{L}{2}) = \kappa_1(0) = \kappa_1(t_1) \ge \kappa_1(t)$ for all $0 \le t \le \frac{L}{2}$. Thus $\kappa_2(t) \ge \kappa_1(t)$. Apply the Schur's arm lemma to

obtain $\text{dist}(\alpha_2(0), \alpha_2(\frac{L}{2})) < \text{dist}(\alpha_1(0), \alpha_1(\frac{L}{2}))$. This contradicts the fact that $\text{dist}(\alpha_2(0), \alpha_2(\frac{L}{2})) = \text{dist}(\alpha_1(0), \alpha_1(\frac{L}{2}))$ since $\alpha_1(0) = \alpha_2(\frac{L}{2})$, $\alpha_1(\frac{L}{2}) = \alpha_2(0)$. Hence α_2 has at least one vertices. But $\kappa_2'(t)$ changes its sign two times, so α has at least 4 vertices.

EXERCISE 27. Verify the inequlaity

$$\text{Length}[\overline{\alpha}] < \text{Length}[\alpha].$$

Hint: It is enough to check that $\int_a^b \sqrt{1 + \overline{f}'(x)^2}\, dx$ is strictly smaller than either of the integrals in the above formula for the length of α.

SOLUTION. We first show that

$$\sqrt{1 + (\frac{f'(x) - g'(x)}{2})^2} < \sqrt{1 + (f'(x))^2} + \sqrt{1 + (g'(x))^2}, \qquad (1)$$

which is equivalent to show that

$$\begin{aligned}
& 1 + (\frac{f'(x) - g'(x)}{2})^2 < 1 + (f'(x))^2 + 1 + (g'(x))^2 \\
& \qquad + 2\sqrt{[1 + (f'(x))^2][1 + (g'(x))^2]} \\
\Leftrightarrow \quad & 4 + (f'(x))^2 + (g'(x))^2 - 2f(x)g'(x) < 8 + 4(f'(x))^2 + 4(g'(x))^2 \\
& \qquad + 8\sqrt{[1 + (f'(x))^2][1 + (g'(x))^2]} \\
\Leftrightarrow \quad & 0 < 3(f'(x))^2 + 3(g'(x))^2 + 2f'(x)g'(x) \\
& \qquad + 4 + 8\sqrt{[1 + (f'(x))^2][1 + (g'(x))^2]} \\
\Leftrightarrow \quad & 0 < 2[(f'(x))^2 + (g'(x))^2] + [f'(x) + g'(x)]^2. \\
& \qquad + 4 + 8\sqrt{[1 + (f'(x))^2][1 + (g'(x))^2]}
\end{aligned}$$

Since the last inequality is obvious, so (1) has been proved.

Integrate both sides of (1) to obtain

$$\int_a^b \sqrt{1 + (\frac{f'(x) - g'(x)}{2})^2}\, dx < \int_a^b \sqrt{1 + (f'(x))^2}\, dx + \int_a^b \sqrt{1 + (g'(x))^2}\, dx.$$

Moreover, note that $f(a) - g(a) \geq 0$ and $f(b) - g(b) \geq 0$. So

$$\int_a^b \sqrt{1 + (\frac{f'(x) - g'(x)}{2})^2}\, dx < [f(a) - g(a)] + \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$
$$+ \int_a^b \sqrt{1 + (g'(x))^2}\, dx + [f(b) - g(b)].$$

Thus

$$\text{Length}[\overline{\alpha}] < \text{Length}[\alpha].$$

EXERCISE 28. Computer the curvature and torsion of the circular helix

$$(r \cos t, r \sin t, ht)$$

where r and h are constants. How does changing the values of r and h effect the curvature and torsion.

SOLUTION. For the first part, the curvature of α at t is

$$\kappa(t) = \frac{|T'(t)|}{|\alpha'(t)|}$$

Note that

$$\begin{aligned}
\alpha(t) &= (r \cos t, r \sin t, ht), \\
\alpha'(t) &= (-r \sin t, r \cos t, h), \\
\left|\alpha'(t)\right| &= \sqrt{r^2 + h^2}.
\end{aligned}$$

so

$$\begin{aligned}
T(t) &= \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{1}{\sqrt{r^2 + h^2}}(-r \sin t, r \cos t, h), \\
T'(t) &= \frac{1}{\sqrt{r^2 + h^2}}(-r \cos t, -r \sin t, 0).
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa(t) &= \frac{|T'(t)|}{|\alpha'(t)|} \\
&= \frac{1}{r^2+h^2}\sqrt{r^2(\cos^2 t+\sin^2 t)} \\
&= \frac{r}{r^2+h^2}.
\end{aligned}$$

The torsion of α is

$$\tau(t) = -\frac{\langle B'(t), N(t)\rangle}{|\alpha'(t)|}.$$

Note that

$$\begin{aligned}
N(t) &= \frac{1}{|T'(t)|}T'(t) \\
&= \frac{\sqrt{r^2+h^2}}{r}\frac{1}{\sqrt{r^2+h^2}}(-r\cos t, -r\sin t, 0) \\
&= (-\cos t, -\sin t, 0) \\
B(t) &= T(t)\times N(t) \\
&= \frac{1}{\sqrt{r^2+h^2}}\left(\begin{vmatrix} r\cos t & h \\ -\sin t & 0 \end{vmatrix}, \begin{vmatrix} h & -r\sin t \\ 0 & -\cos t \end{vmatrix}, \begin{vmatrix} -r\sin t & r\cos t \\ -\cos t & -\sin t \end{vmatrix}\right) \\
&= \frac{1}{\sqrt{r^2+h^2}}(h\sin t, -h\cos t, r) \\
B'(t) &= \frac{1}{\sqrt{r^2+h^2}}(h\cos t, h\sin t, 0).
\end{aligned}$$

Hence

$$\begin{aligned}
\tau(t) &= -\frac{\langle B'(t), N(t)\rangle}{|\alpha'(t)|} \\
&= \frac{h}{r^2+h^2}(\cos^2 t+\sin^2 t) \\
&= \frac{h}{r^2+h^2}.
\end{aligned}$$

For the last part, we first consider effects of changing the values of r and h on the curvature. Let $g(r)$ be the function defined by

$$
\begin{aligned}
g(r) &= \frac{r}{r^2 + h^2} \\
g'(r) &= \frac{h^2 - r^2}{(r^2 + h^2)^2}.
\end{aligned}
$$

Thus

r	0	h	$+\infty$
$g'(r)$	$+$	0	$-$
$\kappa(r)$	$\nearrow$	$\frac{1}{2h}$	$\searrow$

Let $f(h)$ be the function defined by

$$
\begin{aligned}
f(h) &= \frac{r}{r^2 + h^2} \\
f'(h) &= -\frac{2hr}{(r^2 + h^2)^2} < 0 \qquad\qquad \text{if } h, r > 0.
\end{aligned}
$$

Thus if h is increasing, then $\kappa(t)$ is decreasing, and if h is decreasing, then $\kappa(t)$ is increasing .

We now consider effects of changing the values of r and h on the torsion. Let $m(h)$ be the function defined by

$$
\begin{aligned}
m(h) &= \frac{h}{r^2 + h^2} \\
m'(h) &= \frac{r^2 - h^2}{(r^2 + h^2)^2}.
\end{aligned}
$$

Thus

h	0	r	$+\infty$
$m'(h)$	$+$	0	$-$
$\tau(h)$	$\nearrow$	$\frac{1}{2h}$	$\searrow$

Let $n(r)$ be the function defined by

$$
\begin{aligned}
n(r) &= \frac{h}{r^2 + h^2} \\
n'(r) &= -\frac{2hr}{(r^2 + h^2)^2} < 0 \qquad\qquad \text{if } h, r > 0.
\end{aligned}
$$

Thus if r is increasing, then $\tau(t)$ is decreasing, and if r is decreasing, then $\tau(t)$ is increasing.

Exercise 29. By setting $v' = 0$ show that

$$v = T + \frac{\kappa}{\tau} B,$$

and check that v is the desired vector, i.e. $\langle T, v\rangle = const$ and $v' = 0$.

Solution. For the first part, by setting $v' = 0$, we have that v is a fixed point. Then

$$\begin{aligned}\langle T, v\rangle &= a(t)\\ &= a\end{aligned}$$

since $\langle T, B\rangle = \langle T, N\rangle = 0$. By setting $a = 1$, we have that $a' = 0$. Thus

$$\begin{aligned}v' &= T'(t) + b'(t)N(t + b(t)N'(t) + c'(t)B(t) + c(t)B'(t)\\ &= \kappa(t)N(t) + b'(t)N(t) + b(t)[-\kappa(t)T(t) + \tau(t)B(t)] + c'(t)B(t) - c(t)\tau(t)N(t)\end{aligned}$$

since $T'(t) = \kappa(t)N(t)$ and $N'(t) = -\kappa(t)T(t) + \tau(t)B(t)$. Hence

$$v' = -\kappa(t)b(t)T(t) + [\kappa(t) + b'(t) - c(t)\tau(t)]N(t) + [b(t)\tau(t) + c'(t)]B(t).$$

Since $v' = (0, 0, 0)$, so

$$\begin{aligned}-\kappa(t)b(t) &= 0,\\ \kappa(t) + b'(t) - c(t)\tau(t) &= 0,\\ b(t)\tau(t) + c'(t) &= 0,\end{aligned}$$

giving $b(t) = 0$ since $\kappa(t) \neq 0$ and giving $c(t) = \frac{\kappa(t)}{\tau(t)}$. Therefore,

$$v = T + \frac{\kappa}{\tau} B.$$

For the last part, it is clear that

$$\begin{aligned}\langle T, v\rangle &= \left\langle T, T + \frac{\kappa}{\tau} B\right\rangle\\ &= \langle T, T\rangle\\ &= 1\\ &= const\end{aligned}$$

and

$$\begin{aligned} v' &= T' + \frac{\kappa}{\tau}B' \\ &= \kappa N + \frac{\kappa}{\tau}(-\tau N) \\ &= 0 \end{aligned}$$

since $\frac{\kappa}{\tau} = const$, $T' = \kappa N$ and $B' = -\tau N$. Thus v is the desired vector.

EXERCISE 30. Check the converse, that is supposing that the curvature and torsion of some curve satises the above expression, verify whether the curve has to lie on a sphere of radius r.

SOLUTION. Let

$$\rho(t) = \frac{1}{\kappa(t)}.$$

Then

$$\rho'(t) = -\frac{\kappa'(t)}{\kappa^2(t)}.$$

We have that

$$-\frac{\kappa'(t)}{\kappa^2(t)} + \kappa(t)\tau(t)\sqrt{r^2 - \frac{1}{\kappa'(t)}} = 0.$$

This implies that

$$\rho^2(t) + \frac{[\rho'(t)]^2}{\tau^2(t)} = r^2.$$

Differentiate both sides of the last expression to obtain

$$\begin{aligned} & & \frac{d}{dt}(\rho^2(t) + \frac{[\rho'(t)]^2}{\tau^2(t)}) = 0 \\ \iff & & 2\rho(t)\rho'(t) + 2(\frac{\rho'(t)}{\tau(t)})'\frac{\rho'(t)}{\tau(t)} = 0 \\ \iff & & 2\frac{\rho'(t)}{\tau(t)}[\rho(t)\tau(t) + (\frac{\rho'(t)}{\tau(t)})'] = 0, \end{aligned}$$

implying

$$\rho(t)\tau(t) + (\frac{\rho'(t)}{\tau(t)})' = 0 \qquad \text{since } \rho'(t) \neq 0 \qquad (1)$$

On the other hand,

$$\begin{aligned}
&\frac{d}{dt}(\alpha(t) + \rho(t)N(t) + \frac{\rho'(t)}{\tau(t)}B(t)) \\
=&\alpha'(t) + \rho'(t)N(t) + \rho(t)N'(t) + (\frac{\rho'(t)}{\tau(t)})'B(t) + \frac{\rho'(t)}{\tau(t)}B'(t) \\
=&T(t) + \rho'(t)N(t) + \rho(t)[-\kappa(t)T(t) + T(t)B(t)] + (\frac{\rho'(t)}{\tau(t)})'B(t) \\
=&[\rho(t)T(t) + (\frac{\rho'(t)}{\tau(t)})']B(t) && \text{since } \rho(t)\kappa(t) = 1 \\
=&0. && \text{by (1)}
\end{aligned}$$

Thus

$$\alpha(t) + \rho(t)N(t) + \frac{\rho'(t)}{\tau(t)}B(t) = p,$$

which is a fixed point. This implies that

$$\begin{aligned}
\|\alpha(t) - p\| &= \left\| \rho(t)N(t) + \frac{\rho^2(t)}{\tau(t)}B(t) \right\| \\
&= \rho^2(t) + [\frac{\rho^2(t)}{\tau(t)}]^2 && \begin{array}{c}\text{since } N(t) \text{ and } B(t) \\ \text{satisfy } \|N(t)\| = \|B(t)\| = 1 \\ \text{and } \langle N(t), B(t)\rangle = 0.\end{array} \\
&= r^2.
\end{aligned}$$

Therefore $\alpha(t)$ lies on the sphere centered at ρ having radius r.

EXERCISE 31 (Monge Patch). Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$ be a differentiable map. Show that the mapping $X\colon U \to \mathbf{R}^3$, defined by $X(u^1, u^2) := (u^1, u^2, f(u^1, u^2))$ is regular (the pair (X, U) is called a *Monge Patch*).

SOLUTION. The Jacobian of X at p is an 3×2 matrix defined by

$$J_p(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u^1} & \frac{\partial f}{\partial u^2} \end{pmatrix}.$$

But $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. So the rank of $J_p(X)$ is 2. Thus $X(u^1, u^2) := (u^1, u^2, f(u^1, u^2))$ is regular.

EXERCISE 32. Show that $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ is regular at p if and only if

$$\|D_1 f(p) \times D_2 f(p)\| \neq 0.$$

SOLUTION. Assume that $f(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v)) \in \mathbf{R}^3$. Then

$$D_1 f(p) = (\frac{\partial f_1(u,v)}{\partial u}, \frac{\partial f_2(u,v)}{\partial u}, \frac{\partial f_3(u,v)}{\partial u})$$
$$D_2 f(p) = (\frac{\partial f_1(u,v)}{\partial v}, \frac{\partial f_2(u,v)}{\partial v}, \frac{\partial f_3(u,v)}{\partial v})$$

and

$$J_p(f) == \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}.$$

If f is regular, then rank of $J_p(f)$ is 1. This implies that there are two rows of $J_p(f)$ are linealy independent, for example, the 2th row and the 3rd row. Then we have that

$$\begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{vmatrix} \neq 0.$$

Thus

$$\begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix} \neq 0 \qquad\qquad \text{since } \det(A) = \det(A^T).$$

This implies that

$$\|D_1 f(p) \times D_2 f(p)\| = \sqrt{\begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial f_3}{\partial u} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_1}{\partial v} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} \end{vmatrix}^2} > 0.$$

Conversely, if $\|D_1 f(p) \times D_2 f(p)\| \neq 0$, then one of the three terms $\begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix}$, $\begin{vmatrix} \frac{\partial f_3}{\partial u} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_1}{\partial v} \end{vmatrix}$, and $\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} \end{vmatrix}$ has to be different from 0, for example, $\begin{vmatrix} \frac{\partial f_3}{\partial u} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_1}{\partial v} \end{vmatrix} \neq 0$. This implies that $\begin{vmatrix} \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \end{vmatrix} \neq 0$. Thus the rank of $J_p(f)$ is 2, and hence f is regular. Therefore $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ is regular at p if and only if

$$\|D_1 f(p) \times D_2 f(p)\| \neq 0.$$

Exercise 33 (**Surfaces of Revolution**). Let $\alpha\colon I \to \mathbf{R}^2$, $\alpha(t) = (x(t), y(t))$, be a regular simple closed curve. Show that the image of $X\colon I \times R \to \mathbf{R}^3$ given by

$$X(t,\theta) := (x(t)\cos\theta, x(t)\sin\theta, y(t)),$$

is a regular embedded surface.

Solution. $(x(t), y(t))$ is a parametrization of α, given z and $x^2 + y^2 = [(x(t)\cos\theta)^2 + (x(t)\sin\theta)^2] = x^2(t)$, we can determine t uniquely. Thus $X(t,\theta)$ is one-to-one. Since $(x(t), y(t))$ is a parametrization of α, it follows that t is a continuous function of z and of $\sqrt{x^2+y^2}$, and so is a contininious function of (x, y, z). We now need to prove that X^{-1} is continuous. Note that $X\colon U = \{(\theta, t)\colon 0 < \theta < 2\pi, a < t < b\} \to S$ where S is the image of X.

If $\theta \neq \pi$, then

$$\begin{aligned}
\tan\frac{\theta}{2} &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\
&= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} \\
&= \frac{\sin\theta}{1+\cos\frac{\theta}{2}} \\
&= \frac{\frac{y}{x(t)}}{1+\frac{x}{x(t)}} \\
&= \frac{y}{x+\sqrt{x^2+y^2}}.
\end{aligned}$$

Thus

$$\theta = 2\tan^{-1}\frac{y}{x+\sqrt{x^2+y^2}}.$$

Hence if $\theta \neq \pi$, then θ is a continuous function of (x, y, z).

If $\theta = \pi$, then using the fact that

$$\cot\frac{\theta}{2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

and calculations similar to the one above to obtain

$$\theta = 2\cot^{-1}\frac{y}{-x+\sqrt{x^2+y^2}}.$$

Thus θ is a continuous function of (x, y, z). This shows that X^{-1} is a continuous function, and hence the image of $X\colon I \times R \to \mathbf{R}^3$ given by

$$X(t, \theta) := (x(t)\cos\theta, x(t)\sin\theta, y(t)),$$

is a regular embedded surface.

EXERCISE 34. Show that $n\colon \mathbf{S}^2 \to \mathbf{S}^2$, given by $n(p) := p$ is a Gauss map (*Hint:* Define $f\colon \mathbf{R}^3 \to \mathbf{R}$ by $f(p) := \|p\|^2$ and compute its gradient. Note that $\mathbf{S}^2$ is a level set of f. Thus the gradient of f at p must be orthogonal to $\mathbf{S}^2$).

SOLUTION. Define $f\colon \mathbf{R}^3 \to \mathbf{R}$ by $f(p) := \|p\|^2$. Note that $\mathbf{S}^2$ is a level set of f. If $\alpha(t) = (x(t), y(t), z(t))$ is a parametrized curve in $\mathbf{S}^2$, then

$$2xx' + 2yy'^{+=} + 2zz' = 0,$$

which shows that the vector (x, y, z) is normal to the sphere at the point (x, y, z). Restricted to the curve $\alpha(t)$, the normal vector

$$N(t) = (x(t), y(t), z(t))$$

is a vector function of t, and therefore

$$dN(t) = (x'(t), y'(t), z'(t)) = T(t).$$

Since $p \in \mathbf{S}^2$, which is equivalent to $\langle p, p\rangle = 1$, it follows that $\langle p', p\rangle = 0$. Thus $n(p) = p$ is normal to $T_p(S^2)$. and hence $n(p) = p$ is a Gauss map.

EXERCISE 35. Compute the curvature of a sphere of radius r (*Hint:* Use exercise 9).

SOLUTION. The parametrization of the sphere of radius R is

$$x(\theta, \varphi) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta).$$

We have that

$$\begin{aligned}
x_\theta &= (R\cos\theta\cos\varphi, R\cos\theta\sin\varphi, -R\sin\theta)\\
x_\varphi &= (-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0)\\
x_\theta \times x_\varphi &= (R^2\sin^2\theta\cos\varphi, R^2\sin^2\theta\sin\varphi, R^2\sin\theta\cos\theta)\\
\|x_\theta \times x_\varphi\| &= \sqrt{R^4\sin^4\theta(\cos^2\varphi + \sin^2\varphi) + R^4\sin^2\theta\cos^2\theta}\\
&= \sqrt{R^4\sin^2\theta(\sin^2\theta + \cos^2\theta)}\\
&= R^2\sin\theta.
\end{aligned}$$

Therefore,

$$n(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Let S be the shape operator. Then

$$\begin{aligned}
Sx_\theta &= -\partial_\theta n(\theta, \varphi)\\
&= -\partial_\theta(\frac{x(\theta, \varphi)}{R})\\
&= -\frac{1}{R}x_\theta\\
Sx_\varphi &= -\partial_\varphi n(\theta, \varphi)\\
&= -\partial_\varphi(\frac{x(\theta, \varphi)}{R})\\
&= -\frac{1}{R}x_\varphi.
\end{aligned}$$

Hence the shape operator is equal to

$$S = -\frac{1}{R} \cdot I$$

where I is the identity operator. The matrix of S in the basis x_θ, x_φ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}.$$

So, the curvature of the sphere of radius R is

$$\begin{aligned} K &= \begin{vmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{vmatrix} \\ &= \frac{1}{R^2}. \end{aligned}$$

EXERCISE 36. Compute the Gaussian curvature of a surface of revolution, i.e., the surface covered by the patch

$$X(t, \theta) = (x(t)\cos\theta, x(t)\sin\theta, y(t)).$$

Next, letting

$$(x(t), y(t)) = (R + r\cos t, r\sin t),$$

i.e., a circle of radius r centered at $(R, 0)$, compute the curvature of a torus of revolution. Sketch the torus and indicate the regions where the curvature is postive, negative, or zero.

SOLUTION. We have that

$$X_t = (x'(t)\cos\theta, x'(t)\sin\theta, y'(t))$$

$$X_\theta = (-x(t)\sin\theta, x(t)\cos\theta, 0)$$

$$X_t \times X_\theta = (-x(t)y'(t)\cos\theta, -x(t)y'(t)\sin\theta, x(t)x'(t))$$

$$\|X_t \times X_\theta\| = \sqrt{[x(t)y'(t)]^2 + [x(t)x'(t))]^2} = |x(t)|\,\sqrt{[x'(t)]^2 + [y'(t)]^2}$$

$$n(t,\theta) = \frac{1}{|x(t)|\sqrt{[x'(t)]^2 + [y'(t)]^2}}(-x(t)y'(t)\cos\theta, -x(t)y'(t)\sin\theta, x(t)x'(t))$$

$$E = g_{11} = \langle X_t, X_t\rangle = [x'(t)]^2 + [y'(t)]^2$$

$$F = g_{12} = \langle X_t, X_\theta\rangle = 0$$

$$G = g_{22} = \langle X_\theta, X_\theta\rangle = [x(t)]^2$$

$$X_{tt} = (x''(t)\cos\theta, x''(t)\sin\theta, y''(t))$$

$$X_{t\theta} = (-x'(t)\sin\theta, x'(t)\cos\theta, 0)$$

$$X_{\theta\theta} = (-x(t)\cos\theta, -x(t)\sin\theta, 0)$$

$$L = \langle X_{tt}, n(t,\theta)\rangle = \frac{-x(t)x''(t)y'(t) + x(t)x'(t)y''(t)}{|x(t)|\sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

$$M = \langle X_{t\theta}, n(t,\theta)\rangle = 0$$

$$N = \frac{[X(t)]^2 y'(t)}{|x(t)|\sqrt{[x'(t)]^2 + [y'(t)]^2}}.$$

The Gaussian curvature of a surface of revolution

$$\begin{aligned} K(t) &= \frac{\begin{vmatrix} L & M \\ M & N \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} \\ &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{-[x(t)]^2[y'(t)]^2 x''(t) + [x(t)]^3 x'(t)y'(t)y''(t)}{[x(t)]^4\{[x'(t)]^2 + [y'(t)]^2\}^2} \\ &= \frac{-[y'(t)]^2 x''(t) + x'(t)y'(t)y''(t)}{x(t)\{[x'(t)]^2 + [y'(t)]^2\}^2}. \end{aligned}$$

Applying the calculations above for $(x(t), y(t)) = (R + r\cos t, r\sin t)$ to obtain

$$\begin{aligned} x(t) &= R + r\cos t \\ y(t) &= r\sin t \\ x'(t) &= -r\sin t \\ x''(t) &= -r\cos t \\ y'(t) &= r\cos t \\ y''(t) &= -r\sin t \\ [x'(t)]^2 + [y'(t)]^2 &= r^2(\sin^2 t + \cos^2 t) = r^2. \end{aligned}$$

The curvature of a torus of revolution is

$$\begin{aligned} K(t) &= \frac{r^3\cos^3 t + r^3\sin^3 t\cos t}{r^4(R + r\cos t)} \\ &= \frac{\cos t(\cos^2 t + \sin^2 t)}{r(R + r\cos t)} \\ &= \frac{\cos t}{r(R + r\cos t)}. \end{aligned}$$

From this expression, it follows that $K = 0$ along the parallels $t = \pi/2$ and $t = 3\pi/2$; the points of such parallels are therefore parabolic points. In the region of the torus given by $\pi/2 < t < 3\pi/2$, K is negative (notice that $r > 0$ and $R > r$); the points in this region are therefore hyperbolic points. In the region given by $0 < t < \pi/2$ or $3\pi/2 < u < 2\pi$, the curvature is positive and the points are elliptic points (Fig. 3-15).

Exercise 37. Show that $\langle S_p(e_i(p)), e_j(p)\rangle = l_{ij}(0,0)$ (*Hints:* First note that $\langle n(p), e_j(p)\rangle = 0$ for all $p \in V$. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma(0) = e_i(p)$. Define $f\colon (-\epsilon, \epsilon) \to M$ by $f(t) \coloneqq \langle n(\gamma(t)), e_j(\gamma(t))\rangle$, and compute $f'(0)$.)

Solution. First note that $\langle n(p), e_j(p)\rangle = 0$ for all $p \in V$. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma(0) = e_i(p)$. Define $f\colon (-\epsilon, \epsilon) \to M$ by $f(t) \coloneqq n(\gamma(t)), e_j(\gamma(t))$. Since $\langle n(p), e_j(p)\rangle = 0$ for all $p \in V$ and $\gamma(t) \in V$ for $t \in (-\epsilon, \epsilon)$, it follows that

$$f(t) = \langle n(\gamma(t)), e_j(\gamma(t)) \rangle = 0.$$

Differentiate both sides of the last expression to obtain

$$\langle d\, n(\gamma(t)), e_j(\gamma(t)) \rangle + \langle n(\gamma(t)), d\, e_j(\gamma(t)) \rangle = 0.$$

This implies that

$$\left\langle d\, n_{\gamma(t)}(\gamma'(t)), e_j(\gamma(t) \right\rangle = -\left\langle n(\gamma(t)), d\, e_{j_{\gamma(t)}}(\gamma'(t)) \right\rangle.$$

For $t = 0$, then

$$\langle S_p(e_i(p)), e_j(p) \rangle = -\langle n(p), D_{ij}X(0,0) \rangle$$

or

$$\langle S_p(e_i(p)), e_j(p) \rangle = l_{ij}(0,0),$$

as required

EXERCISE 38. Compute the curvature of the graph of $z = ax^2 + by^2$, where a and b are constants. Note how the signs of a and b effect the curvature and shape of the surface. Also note the values of a and b for which the curvature is zero.

SOLUTION. The surface $z = ax^2 + by^2$ is a Monge patch, i.e,

$$X(u_1, u_2) = (u_1, u_2, au_1^2 + bu_2^2),$$

where $f(u_1, u_2) = au_1^2 + bu_2^2$. The Hessian matrix of f is

$$\text{Hess}\,(u_1, u_2) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix},$$

so $\det(\text{Hess}\,(u_1, u_2)) = 4ab$. The gradient of f is

$$\text{grad}\, f = (2au_1, 2bu_2).$$

Thus

$$\text{grad}\, f(0,0) = (0,0).$$

Hence

$$K(p) = \frac{\det(\operatorname{Hess} f(0,0))}{(1 + \|\operatorname{grad} f(0,0)\|^2)^2}$$
$$= 4ab.$$

If a and b have same signs, then $K(p) > 0$. If a and b have opposite signs, then $K(p) < 0$. If $a = 0$ or $b = 0$, then $K(p) = 0$.

EXERCISE 39. Let M be the *Monkey saddle*, i.e., the graph of the equation $z = y^3 - 3yx^2$, and $p := (0,0,0)$. Show that $K(p) = 0$, but M is not locally convex at p.

SOLUTION. For the first part, the surface $z = ax^2 + by^2$ is a Monge patch, i.e,

$$X(u_1, u_2) = (u_1, u_2, u_2^3 - 3u_2u_1^2),$$

where $f(u_1, u_2) = u_2^3 - 3u_2u_1^2$. The Hessian matrix of f is

$$\operatorname{Hess}(u_1, u_2) = \begin{pmatrix} -6u_2 & -6u_1 \\ -6u_2 & 6u_2 \end{pmatrix},$$

so $\det(\operatorname{Hess}(u_1, u_2)) = -36(u_1^2 + u_2^2)$ and $\det(\operatorname{Hess}(0,0)) = 0$. The gradient of f is

$$\operatorname{grad} f = (-6u_1u_2, -3u_2^2 - 3u_1^2).$$

Thus

$$\operatorname{grad} f(0,0) = (0,0).$$

Hence

$$K(p) = \frac{\det(\operatorname{Hess} f(0,0))}{(1 + \|\operatorname{grad} f(0,0)\|^2)^2}$$
$$= 0,$$

as required.

For the last part, we have that

$$X_{u_1}(u_1,u_2) = (1,0,-6u_2u_1)$$
$$X_{u_2}(u_1,u_2) = (0,1,3u_2^2-3u_1^2).$$

Thus

$$X_{u_1}(0,0) = (1,0,0)$$
$$X_{u_2}(0,0) = (0,1,0).$$

The equation of the tangent surface at $(0,0,0)$ is

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} y & z \\ 1 & 0 \end{vmatrix} = 0,$$

that is,

$$z = 0.$$

In a neigborhood of $(0,0,0)$ for $z = u_2(u_2^2 - 3u_1^2)$, then $z > 0$ as $u_2 > 0$ and $u_2^2 > 3u_1^2$; and $z < 0$ as $u_2 > 0$ and $u_2^2 < 3u_1^2$. therefore, in a neighborhood of $(0,0,0)$, then M does not lie on one side of the tangent surface, so MM is not locally convex at $(0,0,0)$

Exercise 40. Show that if $ac - b^2 > 0$, then Q is definite, and if $ac - b^2 < 0$, then Q is not definite. (*Hints*: For the first part, suppose that $x \neq 0$, but $Q(x,y) = 0$. Then $ax^2 + 2bxy + cy^2 = 0$, which yields $a + 2b(x/y) + c(x/y)^2 = 0$. Thus the discriminant of this equation must be positive, which will yield a contradiction. The proof of the second part is similar).

Solution. For the first part, let $ac - b^2 > 0$. Suppose, towards a contradiction, that Q is not definite, i.e, there exists $x \neq 0$, but $Q(x,y) = 0$. Then $ax^2 + 2bxy + cy^2 = 0$, which yields $a + 2b(y/x) + c(y/x)^2 = 0$. Let $t = y/x$. Then the last equation becomes $ct^2 + 2bt + a = 0$. This equation

has solutions so its discriminant must be positive, that is, $b^2 - ac > 0$ which contradicts the fact that $ac - b^2 > 0$. Thus Q is definite.

For the second part, let $ac - b^2 < 0$. Similarly, suppose, towards a contradiction, that Q is definite, i.e, $Q(x, y) \neq 0$ whenver $x \neq 0$. Then $ax^2 + 2bxy + cy^2 \neq 0$ for any $x \neq 0$. This is equivalent to $c(y/x)^2 + 2b(y/x) + a = 0$ has no solution, so the discriminant of this equation must be negative, that is, $b^2 - ac < 0$ which contradicts the fact that $ac - b^2 < 0$. Thus Q is not definite.

Exercise 41. Verify the middle step in the above formula, i.e., show that $dn(D_i X) = D_i(n \circ X)$.

Solution. Let $f(u_1, u_2) = n \circ X(u_1, u_2)$. We have that

$$\begin{aligned} D_i f(u_1, u_2) &= D_i(n \circ X(u_1, u_2)) \\ &= dn(D_i X(u_1, u_2)). \end{aligned}$$

Therefore

$$dn(D_i X) = D_i(n \circ X).$$

Exercise 42. Let $F, G \colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a pair of mappings such that $\langle F, G \rangle = 0$. Prove that $\langle D_i F, G \rangle = -\langle F, D_i G \rangle$.

Solution. Since $\langle F, G \rangle = 0$ so $D_i(\langle F, G \rangle) = 0$. But $D_i(\langle F, G \rangle) = \langle D_i F, G \rangle + \langle F, D_i G \rangle$. Thus $\langle D_i F, G \rangle + \langle F, D_i G \rangle = 0$. Therefore, $\langle D_i F, G \rangle = -\langle F, D_i G \rangle$.

Exercise 43. Show that there exist a patch (U, X) centered at p such that $[g_{ij}(0, 0)]$ is the identity matrix. (*Hint:* Start with a Monge patch with respect to $T_p M$).

Solution. Consider the following maps (assume that $p = (p_1, p_2, f(p_1, p_2))$):
$u_1(t) \colon (-\epsilon, \epsilon) \to U$ given by $u_2(t) = (t, p_2)$,
$c_1 \colon (-\epsilon, \epsilon) \to \mathbb{R}^3$ given by $c_1(t) = \frac{1}{\sqrt{1+(f_u')^2}} X \circ u_1(t) = \frac{1}{\sqrt{1+(f_u')^2}}(t, p_2, f(t, p_2))$,
$u_2(t) \colon (-\epsilon, \epsilon) \to U$ given by $u_2(t) = (p_1, t + p_2)$,
$c_2 \colon (-\epsilon, \epsilon) \to \mathbb{R}^3$ given by $c_2(t) = \frac{1}{\sqrt{1+(f_v')^2}} X \circ u_2(t) = \frac{1}{\sqrt{1+(f_v')^2}}(p_1, t + p_2, f(p_1, t + p_2))$

$\gamma(t) \colon (-\epsilon, \epsilon) \to \mathbb{R}^3$ given by $\gamma(t) = \begin{cases} c_1(t) & \text{if } t < 0, \\ c_2(t) & \text{if } t \geq 0. \end{cases}$

It is clear that $\gamma(0) = (p_1, p_2, f(p_1, p_2)) = p$. If $t < 0$, then $D_1X \circ \gamma(t) = \frac{1}{\sqrt{1+(f'_u)^2}}(1, 0, f'_u)$ and $D_2X \circ \gamma(t) = (0, 0, 0)$. If $t \geq 0$, then $D_1X \circ \gamma(t) = (0, 0, 0)$ and $D_2X \circ \gamma(t) = \frac{1}{\sqrt{1+(f'_v)^2}}(0, 1, f'_v)$. Thus

$$\langle D_iX \circ \gamma(t), D_jX \circ \gamma(t)\rangle = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Therefore $[g_{ij}(0, 0)]$ is the identity matrix.

Exercise 44. Show that $k_v(p)$ does not depend on γ.

Solution. We have that

$$\begin{aligned} k_v(p) &= \big\langle n(p), \gamma''(0)\big\rangle \\ &= \big\langle (n \circ \gamma)(0), \gamma''(0)\big\rangle \\ &= -\big\langle (n \circ \gamma)'(0), \gamma'(0)\big\rangle \\ &= -\langle d\,n_p(v), v\rangle \\ &= \langle S_p(v), v\rangle \\ &= \mathrm{II}_p(v). \end{aligned}$$

Therefore $k_v(p)$ does not depend on γ.

Exercise 45. Show that II_p is symmetric, i.e., $\mathrm{II}_p(v, w) = \mathrm{II}_p(w, v)$ for all $v, w \in T_pM$.

Solution. Let $\gamma(t) = x(u(t), v(t))$, where $x(u, v)$ is a parametrization of S at p and $\{x_u, x_v\}$ is the associated basis of $T_p(S)$. Note that $\gamma(0) = p$. We have that

$$\begin{aligned} d\,N_p(\gamma'(0)) &= d\,N_p(x_u u'(0) + x_v v'(0)) \\ &= \frac{dN}{dt}(u(t), v(t))|_{t=0} \\ &= N_u u'(0) + N_v v'(0). \end{aligned}$$

In particular, $d\,N_p(x_u) = N_u$, $d\,N_p(x_v) = N_v$ so to prove that $d\,N_p$ is a self-adjoint, we need only show that $\langle N_u, x_v\rangle = \langle N_v, x_u\rangle$. Take derivative both sides of $\langle N, x_u\rangle = 0$ and $\langle N, x_v\rangle = 0$ with respect to v and u, respectively, to obtain

$$\langle N_v, x_u \rangle + \langle N, x_{uv} \rangle = 0$$
$$\langle N_u, x_v \rangle + \langle N, x_{vu} \rangle = 0$$

Thus $\langle N_u, x_v \rangle = -\langle N, x_{vu} \rangle = -\langle N, x_{uv} \rangle = \langle N_v, x_u \rangle$ (since $x_{uv} = x_{vu}$).

Hence $d\,N_p\colon T_p(S) \to T_p(S)$ is a self-adjoint linear map, so the shape operator $S_p = -d\,N_p$ is also a self-adjoint linear map. Therefore $\langle S_p(u), v \rangle = \langle u, S_p(v) \rangle = \langle S_p(v), u \rangle$. This implies that $\mathrm{II}_p(u,v) = \mathrm{II}_p(v,u)$ (Note that $\mathrm{II}_p(u,v) = \langle S_p(u), v \rangle$).

Exercise 46. Verify the above claim, and show that minimum and maximum values of II_p are λ_1 and λ_1 respectively. Thus $k_1(p) = \lambda_1$, and $k_2(p) = \lambda_2$.

Solution. We have that

$$v(\theta) = \cos\theta \cdot e_1 + \sin\theta \cdot e_2.$$

Then

$$\begin{aligned} S_p(v(\theta)) &= S_p(\cos\theta \cdot e_1 + \sin\theta \cdot e_2) \\ &= S_p(\cos\theta \cdot e_1) + S_p(\sin\theta \cdot e_2) && \text{since } S_p \text{ is linear} \\ &= \cos\theta \cdot S_p(e_1) + \sin\theta \cdot S_p(e_2) && \text{since } S_p \text{ is linear} \\ &= \lambda_1 \cos\theta \cdot e_1 + \lambda_2 \sin\theta \cdot e_2 && \text{since } S_p(e_1) = \lambda_1 e_1 \text{ and } S_p(e_2) = \lambda_2 e_2. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{II}_p(v(\theta), v(\theta)) &= \langle S_p(v(\theta)), v(\theta) \rangle \\ &= \langle \lambda_1 \cos\theta \cdot e_1 + \lambda_2 \sin\theta \cdot e_2, \cos\theta \cdot e_1 + \sin\theta \cdot e_2 \rangle \\ &= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta \end{aligned}$$

since

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

By the hypothesis that $\lambda_1 \leq \lambda_2$, we have that

$$\begin{aligned} \mathrm{II}_p(v(\theta), v(\theta)) &= \cos^2\theta \cdot \lambda_1 + \sin^2\theta \cdot \lambda_2 \\ &\leq \cos^2\theta \cdot \lambda_2 + \sin^2\theta \cdot \lambda_2 \\ &= (\cos^2\theta + \sin^2\theta) \cdot \lambda_2 \\ &= \lambda_2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{II}_p(v(\theta), v(\theta)) &= \cos^2\theta \cdot \lambda_1 + \sin^2\theta \cdot \lambda_2 \\ &\geq \cos^2\theta \cdot \lambda_1 + \sin^2\theta \cdot \lambda_1 \\ &= (\cos^2\theta + \sin^2\theta) \cdot \lambda_2 \\ &= \lambda_1. \end{aligned}$$

Thus $\lambda_1 \leq \mathrm{II}_p(v(\theta), v(\theta)) \leq \lambda_2$ for all $v(\theta) \in UT_p(M)$, and it is clear that $\mathrm{II}_p(v(0), v(0)) = \lambda_1$ and $\mathrm{II}_p(v(\frac{\pi}{2}), v(\frac{\pi}{2})) = \lambda_2$. Hence minimum and maximum values of II_p are λ_1 and λ_1 respectively.

Moreover, we have that

$$\begin{aligned} \langle S_p(v), v\rangle &= -\langle d\, n_p(v), v\rangle \\ &= -\big\langle (n \circ \gamma)'(0), \gamma'(0)\big\rangle \\ &= \big\langle (n \circ \gamma)(0), \gamma''(0)\big\rangle \\ &= \big\langle n(p), \gamma''(0)\big\rangle \\ &= k_v(p). \end{aligned}$$

Thus $k_v(p) = \langle S_p(v), v\rangle = \mathrm{II}_p(v, v)$. Hence $k_1 = \min_v k_v(p) = \min_v \mathrm{II}_p(v, v) = \lambda_1$ and $k_2 = \max_v k_v(p) = \max_v \mathrm{II}_p(v, v) = \lambda_2$.

EXERCISE 47. What is $\det(l_i^j)$ equal to?

SOLUTION. Since $(l_i^j) = (l_{ij})(g_{ij})^{-1}$, it follows that

$$\begin{aligned} \det(l_i^j) &= \det(l_{ij}) \det[(g_{ij})^{-1}] \\ &= \frac{\det(l_{ij})}{\det(g_{ij})} \qquad\qquad \text{since } \det[(g_{ij})^{-1}] = \frac{1}{\det(g_{ij})} \\ &= \frac{\det(\langle N, X_{ij}\rangle)}{\det(\langle X_i, X_j\rangle)}. \end{aligned}$$

Thus $\det(l_i^j) = \frac{\det(\langle N, X_{ij}\rangle)}{\det(\langle X_i, X_j\rangle)} = ?$.

EXERCISE 48. Show that $N_i = -dn(X_i) = S(X_i)$.

SOLUTION. The matrix of $dn(X_i)$ is

$$\begin{aligned}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22}\end{pmatrix} &= -\begin{pmatrix} e & f \\ f & g\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1} \\ &= -\left(l_{ij}\right)\left(g_{ij}\right)^{-1} \\ &= -\left(l_i^j\right).\end{aligned}$$

Thus $N_i = -l_i^1 X_1 - l_i^2 X_2 = dn(X_i) = -S(X_i)$.

EXERCISE 49. Compute the Christoffel symbols of a surface of revolution.

SOLUTION. Consider a surface of revolution parametrized by

$$\begin{aligned} x(u,v) &= (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)) \\ &\quad 0 < u < 2\pi, \qquad a < v < b, \qquad \psi(v) \neq 0. \end{aligned}$$

We have that

$$X_1 = (-\varphi(v)\sin u, \varphi(v)\cos u, 0)$$

$$X_2 = (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v))$$

$$X_{11} = (-\varphi(v)\cos u, -\varphi(v)\sin u, 0)$$

$$X_{12} = (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0)$$

$$X_{21} = (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0)$$

$$X_{22} = (\varphi''(v)\cos u, \varphi''(v)\sin u, \psi''(v))$$

$$g_{11} = \langle X_1, X_1\rangle = \varphi^2(v)\sin^2 u + \varphi^2(v)\cos^2 u = \varphi^2(v)$$

$$g_{12} = \langle X_1, X_2\rangle = 0 = g_{21}$$

and

$$\begin{aligned} g_{22} &= [\varphi'(v)]^2\cos^2 u + [\varphi'(v)]^2\sin^2 u + [\psi'(v)]^2 \\ &= [\varphi'(v)]^2 + [\psi'(v)]^2 = 1 \end{aligned}$$

since assume that the rotating curve is parametrized by arc-length.Thus

$$\begin{aligned} (g_{ij}) &= \begin{pmatrix} \varphi^2 & 0 \\ 0 & 1 \end{pmatrix}, \\ (g^{ij}) &= (g_{ij})^{-1} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}. \end{aligned}$$

- Γ_{11}^1, Γ_{11}^2: We have that

$$\begin{aligned} &\langle X_{11}, X_1\rangle = 0, \\ &\langle X_{11}, X_2\rangle = -\varphi(v)\varphi'(v) = -\varphi\varphi'. \end{aligned}$$

Thus $\begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}\begin{pmatrix} 0 \\ -\varphi\varphi' \end{pmatrix} = \begin{pmatrix} 0 \\ -\varphi\varphi' \end{pmatrix}$. Hence $\Gamma_{11}^1 = 0$, $\Gamma_{11}^2 = -\varphi\varphi'$.

- Γ_{12}^1, Γ_{12}^2, Γ_{21}^1, Γ_{21}^2: We have that

$$\begin{aligned} &\langle X_{12}, X_1\rangle = \varphi\varphi', \\ &\langle X_{12}, X_2\rangle = 0. \end{aligned}$$

Thus $\begin{pmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}\begin{pmatrix} \varphi\varphi' \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\varphi'}{\varphi} \\ 0 \end{pmatrix}$. Hence $\Gamma_{12}^1 = \frac{\varphi'}{\varphi} = \Gamma_{21}^1$, $\Gamma_{12}^2 = 0 = \Gamma_{21}^2$ (Note that $\Gamma_{ij}^k = \Gamma_{ji}^k$).

- Γ_{22}^1, Γ_{22}^2: We have that

$$\begin{aligned} &\langle X_{22}, X_1\rangle = 0, \\ &\langle X_{22}, X_2\rangle = \varphi'\varphi'' + \psi'\psi''. \end{aligned}$$

Thus $\begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi'\varphi'' + \psi'\psi'' \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'\varphi'' + \psi'\psi'' \end{pmatrix}$. Hence $\Gamma_{22}^1 = 0$, $\Gamma_{22}^2 = \varphi'\varphi'' + \psi'\psi''$.

EXERCISE 50. Show that if $M = \mathbf{R}^2$, then $R^l_{ijk} = 0$ for all $1 \le i, l, j, k \le 2$ both intrinsically and extrinsically.

SOLUTION. We first show the claim extrinsically. If $M = \mathbf{R}^2$, then $X(u, v) = (u, v, 0)$. Thus $X_{ij} = (0, 0, 0)$ for $i, j \in \{1, 2\}$. Hence $l_{ij} = \langle N, X_{ij}\rangle = 0$ for $i, j \in \{1, 2\}$. Therefore $R^l_{ijk} = l_{ik}l^l_j - l_{ij}l^l_k = 0$ for all $1 \le i, l, j, k \le 2$.

We next show the claim intrinsically. We have that

$$g_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Thus

$$\begin{aligned}\Gamma^k_{ij} &= \sum_{l=1}^{2} \frac{1}{2}[(g_{li})_j + (g_{jl})_i - (g_{ij})_l]g^{lk} \\ &= 0 \qquad\qquad \text{for all } 1 \le i, l, j, k \le 2.\end{aligned}$$

Hence

$$\begin{aligned}R^l_{ijk} &= (\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2}(\Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}) \\ &= 0 \qquad\qquad \text{for all } 1 \le i, l, j, k \le 2.\end{aligned}$$

EXERCISE 51. Compute the Riemann curvature tensor for S^2 both intrinsically and extrinsically.

SOLUTION. It is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi); 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $x : V \to \mathbf{R}^3$ be given by

$$x(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

By the convention that $\varphi(\theta) = \sin\theta$, $\psi(\theta) = \cos\theta$, we have that

$$X(\varphi, \theta) = (\varphi(\theta)\cos\varphi, \varphi(\theta)\sin\varphi, \psi(\theta))$$

where $0 < \varphi < 2\pi$, $0 < \theta < \pi$, $\varphi(\theta) > 0$. Since S^2 is a surface of revolution and $X(\varphi, \theta)$ is a parametrization of S^2.

We first show the claim extrinsically. Using Exercise 3 of Lecture Notes 12 to obtain

$$\begin{aligned} g_{11} &= \sin^2\theta, \\ g_{12} &= g_{21} = 0, \\ g_{22} &= 1. \end{aligned}$$

Thus

$$\begin{aligned} (g_{ij}) &= \begin{pmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{pmatrix}, \\ (g^{ij}) &= (g_{ij})^{-1} = \begin{pmatrix} \frac{1}{\sin^2\theta} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We have that

$$\begin{aligned} N &= \frac{X_1 \times X_2}{\|X_1 \times X_2\|} = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, -\cos\theta), \\ l_{11} &= \langle N, X_{11}\rangle = \sin^2\theta, \\ l_{12} &= l_{21} = \langle N, X_{12}\rangle = 0, \\ l_{22} &= \langle N, X_{22}\rangle = 1, \\ (l_i^j) &= (l_{ij})(g^{ij}) = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sin^2\theta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence

$$l_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus

$$R^l_{iii} = 0 \qquad\qquad \text{for } i = 1, 2, l \neq i,$$

$$R^l_{111} = \sin^2\theta - \sin^2\theta = 0,$$

$$R^2_{222} = 1 \cdot 1 - 1 \cdot 1 = 0,$$

$$R^1_{112} = l_{12}l^1_1 - l_{11}l^1_2 = 0,$$

$$R^1_{121} = l_{11}l^1_2 - l_{12}l^1_1 = 0,$$

$$R^1_{211} = l_{21}l^1_1 - l_{21}l^1_1 = 0,$$

$$R^2_{112} = l_{12}l^2_1 - l_{11}l^2_2 = -\sin^2\theta,$$

$$R^2_{121} = l_{11}l^2_2 - l_{12}l^2_1 = \sin^2\theta,$$

$$R^2_{211} = l_{21}l^2_1 - l_{21}l^2_1 = 0,$$

$$R^1_{122} = l_{12}l^1_2 - l_{12}l^1_2 = 0,$$

$$R^1_{212} = l_{22}l^1_1 - l_{21}l^1_2 = 1,$$

$$R^1_{221} = l_{21}l^1_2 - l_{22}l^1_1 = -1,$$

$$R^2_{122} = l_{12}l^2_2 - l_{12}l^2_2 = 0,$$

$$R^2_{212} = l_{22}l^2_1 - l_{21}l^2_2 = 0,$$

$$R^2_{221} = l_{21}l^2_2 - l_{22}l^2_1 = 0.$$

We next show the claim intrinsically. Using Exercise 3 of Lecture Notes 12 to obtain

$$\begin{aligned}
&\Gamma^1_{11} = 0,\\
&\Gamma^2_{11} = -\sin\theta\cos\theta,\\
&\Gamma^1_{12} = \Gamma^1_{21} = \frac{\cos\theta}{\sin\theta},\\
&\Gamma^2_{12} = \Gamma^2_{21} = 0,\\
&\Gamma^1_{22} = 0,\\
&\Gamma^2_{22} = \varphi'\varphi'' + \psi'\psi'' = \cos\theta(-\sin\theta) + (-\sin\theta)(-\cos\theta) = 0.
\end{aligned}$$

Thus

$$\begin{aligned} R^l_{iii} &= (\Gamma^l_{ii})_i - (\Gamma^l_{ii})_i + \Gamma^1_{ii}\Gamma^l_{1i} - \Gamma^1_{ii}\Gamma^l_{1i} \\ &\quad + \Gamma^2_{ii}\Gamma^l_{2i} - \Gamma^2_{ii}\Gamma^l_{2i} \\ &= 0, \qquad\qquad \text{for } i, l \in \{1, 2\}. \end{aligned}$$

$$\begin{aligned} R^1_{112} &= (\Gamma^1_{12})_1 - (\Gamma^1_{11})_2 + \Gamma^1_{12}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{12} \\ &\quad + \Gamma^2_{12}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{22} \\ &= 0, \end{aligned}$$

$$\begin{aligned} R^1_{121} &= (\Gamma^1_{11})_2 - (\Gamma^1_{12})_1 + \Gamma^1_{11}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{11} \\ &\quad + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{21} \\ &= 0, \end{aligned}$$

$$\begin{aligned} R^1_{211} &= (\Gamma^1_{21})_1 - (\Gamma^1_{21})_1 + \Gamma^1_{21}\Gamma^1_{11} - \Gamma^1_{21}\Gamma^1_{11} \\ &\quad + \Gamma^2_{21}\Gamma^1_{21} - \Gamma^2_{21}\Gamma^1_{21} \\ &= 0, \end{aligned}$$

$$\begin{aligned} R^2_{112} &= (\Gamma^2_{12})_1 - (\Gamma^2_{11})_2 + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12} \\ &\quad + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22} \\ &= \cos^2\theta - \sin^2\theta - \cos^2\theta = -\sin^2\theta, \end{aligned}$$

$$\begin{aligned} R^2_{121} &= (\Gamma^2_{11})_2 - (\Gamma^2_{12})_1 + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} \\ &\quad + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{21} \\ &= \frac{d}{d\theta}(-\sin\theta\cos\theta) - \frac{\cos\theta}{\sin\theta}(-\sin\theta\cos\theta) \\ &= -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta \end{aligned}$$

$$
\begin{aligned}
R^2_{211} &= (\Gamma^2_{21})_1 - (\Gamma^2_{21})_1 + \Gamma^1_{21}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{11} \\
&\quad + \Gamma^2_{21}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^2_{21} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
R^1_{122} &= (\Gamma^1_{12})_2 - (\Gamma^1_{12})_2 + \Gamma^1_{12}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} \\
&\quad + \Gamma^2_{12}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{22} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
R^1_{212} &= (\Gamma^1_{22})_1 - (\Gamma^1_{21})_2 + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^1_{21}\Gamma^1_{12} \\
&\quad + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^2_{21}\Gamma^1_{22} \\
&= -\frac{d}{d\theta}(\frac{\cos\theta}{\sin\theta}) - \frac{\cos^2\theta}{\sin^2\theta} \\
&= -\frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} = 1,
\end{aligned}
$$

$$
\begin{aligned}
R^1_{221} &= (\Gamma^1_{21})_2 - (\Gamma^1_{22})_1 + \Gamma^1_{21}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^1_{11} \\
&\quad + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^2_{22}\Gamma^1_{21} \\
&= \frac{d}{d\theta}(\frac{\cos\theta}{\sin\theta}) + \frac{\cos^2\theta}{\sin^2\theta} \\
&= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = -1,
\end{aligned}
$$

$$
\begin{aligned}
R^2_{122} &= (\Gamma^2_{12})_2 - (\Gamma^2_{12})_2 + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{12} \\
&\quad + \Gamma^2_{12}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{22} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
R^2_{212} &= (\Gamma^2_{22})_1 - (\Gamma^2_{21})_2 + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{12} \\
&\quad + \Gamma^2_{22}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^2_{22} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
R^2_{221} &= (\Gamma^2_{21})_2 - (\Gamma^2_{22})_1 + \Gamma^1_{21}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} \\
&\quad + \Gamma^2_{21}\Gamma^2_{22} - \Gamma^2_{22}\Gamma^2_{21} \\
&= 0.
\end{aligned}
$$

EXERCISE 52. Show that if Z is a tangent vectorfield of A and $f\colon A \to \mathbf{R}$ is a function, then

$$\overline{\nabla}_{W+Z}V = \overline{\nabla}_W V + \overline{\nabla}_Z V, \qquad \text{and} \qquad \overline{\nabla}_{fW}V = f\overline{\nabla}_W V.$$

Further if $Z\colon A \to \mathbf{R}^n$ is any vectorfield, then

$$\overline{\nabla}_W(V+Z) = \overline{\nabla}_W V + \overline{\nabla}_Z Z, \quad \text{and} \quad \overline{\nabla}_W(fV) = (Wf)V + f\overline{\nabla}_W V.$$

SOLUTION. Assume that the curve $\gamma\colon(-\epsilon,\epsilon) \to A \subset \mathbf{R}^n$ with $\gamma(0) = p$, $\gamma'(0) = W$.Then

$$\begin{aligned}\overline{\nabla}_W V &= (V \circ \gamma)'(0)\\ &= ((V^1 \circ \gamma^1)(0), (V^2 \circ \gamma^2)(0), \dots, (V^n \circ \gamma^n)(0)),\end{aligned}$$

where $\gamma = (\gamma^1, \gamma^2, \dots, \gamma^n)$ and γ^k are the component functions of γ and $V = (V^1, V^2, \dots, V^n)$ with V^i are the component functions of V. Moreover,

$$((V^1 \circ \gamma^1)'(0), (V^2 \circ \gamma^2)'(0), \dots, (V^n \circ \gamma^n)'(0)) = (WV^1, WV^2, \dots WV^n).$$

Thus

$$\overline{\nabla}_W V = (WV^1, WV^2, \dots, WV^n). \tag{1}$$

Using (1) to obtain

$$\begin{aligned}&\overline{\nabla}_{W+Z}V\\ &= ((W+Z)V^1, (W+Z)V^2, \dots, (W+Z)V^n)\\ &= (WV^1 + ZV^1, WV^2 + ZV^2, \dots .WV^n + ZV^n)\\ &= (WV^1, WV^2, \dots, WV^n) + (ZV^1, ZV^2, \dots, ZV^n)\\ &= \overline{\nabla}_W V + \overline{\nabla}_Z V,\end{aligned}$$

$$\begin{aligned}(W+Z)f &= \langle W+Z, \mathrm{grad} f\rangle\\ &= \langle W, \mathrm{grad} f\rangle + \langle Z, \mathrm{grad} f\rangle\\ &= Wf + Zf \text{ for all } f\colon A \to \mathbf{R}\end{aligned}$$

as required. Again, applying (1) to obtain

$$\begin{aligned}\overline{\nabla}_{fW}V &= (fWV^1, fWV^2, \dots fWV^n)\\ &= f(WV^1, WV^2, \dots, WV^n)\\ &= f\overline{\nabla}_W V,\end{aligned}$$

as required. Further, we have that

$$\begin{aligned}\overline{\nabla}_W(V+Z) &= [(V+Z)\circ\gamma]'(0)\\ &= (V\circ\gamma)'(0)+(Z\circ\gamma)'(0)\\ &= \overline{\nabla}_W V+\overline{\nabla}_W Z,\end{aligned}$$

as required. Moreover, we have that

$$\begin{aligned}\overline{\nabla}_W(fV) &= (WfV^1, WfV^2, \ldots, WfV^n)\\ &= ((fV^1\circ\gamma)'(0), (fV^2\circ\gamma)'(0), \ldots, (fV^n\circ\gamma)'(0)),\end{aligned} \qquad (2)$$

and

$$(Wf)V = (WfV^1, WfV^2, \ldots, WfV^n),$$

and

$$f\overline{\nabla}_W W = (fWV^1, fWV^2, \ldots, fWV^n).$$

Therefore,

$$(Wf)V + f\overline{\nabla}_W V = (WfV^1 + fWV^1, WfV^2 + fWV^2, \ldots, WfV^n + fWV^n). \quad (3)$$

But by Leibnitz rule,

$$(fV^i\circ\gamma)'(0) = WfV^i + fWV^i \qquad \text{for } i = 1, 2, \ldots, n. \qquad (4)$$

It follows from (2), (3) and (4) that

$$\overline{\nabla}_W(fV) = (Wf)V + f\overline{\nabla}_W V,$$

as required.

Exercise 53. Note that if V and W are a pair of vectorfields on A then $\langle V, W\rangle : A \to \mathbf{R}$ defined by $\langle V, W\rangle_p := \langle V_p, W_p\rangle$ is a function on A, and show that

$$Z\langle V, W\rangle = \left\langle \overline{\nabla}_Z V, W\right\rangle + \left\langle V, \overline{\nabla}_Z W\right\rangle.$$

SOLUTION. Assume that the curve $\gamma\colon (-\epsilon, \epsilon) \to A \subset \mathbf{R}^n$ with $\gamma(0) = p$, $\gamma'(0) = Z$.We have that

$$
\begin{aligned}
&Z \langle V, W\rangle \\
&= (\langle V, W\rangle \circ \gamma)'(0) \\
&= (\langle V \circ \gamma, W \circ \gamma\rangle)'(0) \\
&= \big\langle (V \circ \gamma)'(0), W \circ \gamma(0)\big\rangle + \big\langle (V \circ \gamma)(0), (W \circ \gamma)'(0)\big\rangle \quad \text{since } (\langle u, v\rangle)' = \langle u', v\rangle + \langle u, v'\rangle \\
&= \big\langle \overline{\nabla}_Z V, W\big\rangle + \big\langle V, \overline{\nabla}_Z W\big\rangle,
\end{aligned}
$$

as required.

EXERCISE 54. Show that $\overline{R} \equiv 0$.

SOLUTION. For every function $f\colon A \to \mathbf{R}$, we have that $Wf = (f \circ \gamma)'(0) = \sum_{i=1}^n D_i f(p)(\gamma^i)'(0) = \sum_{i=1}^n D_i f(p) W^i = \langle W, \mathrm{grad} f\rangle$ where $\gamma = (\gamma^1, \gamma^2, \ldots, \gamma^n)$, $\gamma(0) = p$, $\gamma'(0) = W$ and $W = (W^1, W^2, \ldots, W^n)$.

Next, we define $\mathrm{Hess} f(V, W) = \big\langle V, \overline{\nabla}_W \mathrm{grad} f\big\rangle$, and we shall show that $\mathrm{Hess} f(V, W) = \mathrm{Hess} f(W, V)$. Indeed,

$$
\begin{aligned}
\mathrm{Hess} f(V, W) &= \big\langle V, \overline{\nabla}_W \mathrm{grad} f\big\rangle \\
&= \langle V, (W(D_1 f), W(D_2 f), \ldots, W(D_n f))\rangle \\
&= \langle V, (\langle W, \mathrm{grad} D_1 f\rangle, \langle W, \mathrm{grad} D_2 f\rangle, \ldots, \langle W, \mathrm{grad} D_n f\rangle)\rangle \\
&= \sum_{i=1}^n V^i \langle W, \mathrm{grad} D_i f\rangle \\
&= \sum_{i=1}^n V^i (\sum_{j=1}^n W^j D_{ji} f) \\
&= \sum_{i,j=1}^n V^i W^j D_{ij} f \\
&= \langle W, (\langle V, \mathrm{grad} D_1 f\rangle, \langle V, \mathrm{grad} D_2 f\rangle, \ldots, \langle V, \mathrm{grad} D_n f\rangle)\rangle \\
&= \big\langle W, \overline{\nabla}_V \mathrm{grad} f\big\rangle \\
&= \mathrm{Hess} f(W, V).
\end{aligned}
$$

Using all facts above to obtain that

$$
\begin{aligned}
&V(Wf) - W(Vf) \\
&= V\langle W, \mathrm{grad} f\rangle - W\langle V, \mathrm{grad} f\rangle \\
&= \left\langle \overline{\nabla}_V W, \mathrm{grad} f\right\rangle + \left\langle W, \overline{\nabla}_V \mathrm{grad} f\right\rangle && \text{by Exercise 4 of Lecture Notes 13} \\
&\qquad - \left\langle \overline{\nabla}_W V, \mathrm{grad} f\right\rangle - \left\langle V, \overline{\nabla}_W \mathrm{grad} f\right\rangle && \text{solved above} \\
&= \left\langle \overline{\nabla}_V W - \overline{\nabla}_W V, \mathrm{grad} f\right\rangle \\
&\qquad + \mathrm{Hess} f(W,V) - \mathrm{Hess} f(V,W) \\
&= \langle [V,W], \mathrm{grad} f\rangle && \begin{array}{c}\text{by the fact that} \\ \mathrm{Hess} f(V,W) = \mathrm{Hess} f(W,V) \text{ proved above}\end{array} \\
&= [W,W] f.
\end{aligned}
$$

Thus

$$V(Wf) - W(Vf) = [W,W]f.$$

Let $Z = (Z^1.Z^2, \ldots, Z^n)$. Then

$$V(WZ^i) - W(VZ^i) = [V,W]Z^i$$

for $i = 1, 2, \ldots n$. Thus

$$
\begin{aligned}
&(V(WZ^1), V(WZ^2), \ldots, V(WZ^n)) - (W(VZ^1), W(VZ^2), \ldots, W(VZ^n)) \\
=&([V,W]Z^1, [V,W]Z^2, \ldots, [V,W]Z^n).
\end{aligned}
$$

Using the fact that $\overline{\nabla}_W Z = (WZ^1, WZ^2, \ldots, WZ^n)$ to obtain

$$\overline{\nabla}_V \overline{\nabla}_W Z - \overline{\nabla}_W \overline{\nabla}_V Z = \overline{\nabla}_{[V,W]} Z.$$

Therefore,

$$
\begin{aligned}
\overline{R}(V,W)Z &= \overline{\nabla}_V \overline{\nabla}_W Z - \overline{\nabla}_W \overline{\nabla}_V Z - \overline{\nabla}_{[V,W]} Z \\
&= 0,
\end{aligned}
$$

and so $\overline{R} \equiv 0$ as required.

Exercise 55. Show that, with respect to tangent vectorfields on M, ∇ satisfies all the properties which were listed for $\overline{\nabla}$ in Exercises 52 and 53.

Solution. Let n be a unit normal vector to T_pM at p.
Firstly, we have that

$$
\begin{aligned}
&\nabla_{W+Z}V \\
&= (\overline{\nabla}_{W+Z}V)^T \\
&= (\overline{\nabla}_W V + \overline{\nabla}_Z V) - \left\langle \overline{\nabla}_W V + \overline{\nabla}_Z V, n \right\rangle n && \text{since } \overline{\nabla}_{W+Z}V = \overline{\nabla}_W V + \overline{\nabla}_Z V \\
&= [\overline{\nabla}_W V - \left\langle \overline{\nabla}_W V, n \right\rangle n] + [\overline{\nabla}_Z V - \left\langle \overline{\nabla}_Z V, n \right\rangle n] \\
&= [\overline{\nabla}_W V - (\overline{\nabla}_W)^{\perp}] + [\overline{\nabla}_Z V - (\overline{\nabla}_Z)^{\perp}] \\
&= (\overline{\nabla}_W V)^T + (\overline{\nabla}_Z V)^T \\
&= \nabla_W V + \nabla_Z V.
\end{aligned}
$$

Thus $\nabla_{W+Z}V = \nabla_W V + \nabla_Z V$.

Secondly,

$$
\begin{aligned}
\nabla_{fM}V &= (\overline{\nabla}_{fM}V)^T \\
&= \overline{\nabla}_{fM}V - \left\langle \overline{\nabla}_{fM}V, n \right\rangle n \\
&= f\overline{\nabla}_M V - f\left\langle \overline{\nabla}_M V, n \right\rangle n && \text{since } \overline{\nabla}_{fM}V = f\overline{\nabla}_M V \\
&= f[\overline{\nabla}_M V - \left\langle \overline{\nabla}_M V, n \right\rangle n] \\
&= f[\overline{\nabla}_M V - (\overline{\nabla}_M V)^{\perp}] \\
&= f(\overline{\nabla}_M V)^T \\
&= f\nabla_M V.
\end{aligned}
$$

Thus $\nabla_{fM}V = f\nabla_M V$.

Thirdly,

$$
\begin{aligned}
&\nabla_W(V+Z) \\
&= \overline{\nabla}_W(V+Z) - \left\langle \overline{\nabla}_W(V+Z), n \right\rangle n \\
&= \overline{\nabla}_W V + \overline{\nabla}_W Z - \left\langle \overline{\nabla}_W V, n \right\rangle n - \left\langle \overline{\nabla}_W Z, n \right\rangle n && \text{since } \overline{\nabla}_W(V+Z) = \overline{\nabla}_W V + \overline{\nabla}_W Z \\
&= (\overline{\nabla}_W V - \left\langle \overline{\nabla}_W V, n \right\rangle n) + (\overline{\nabla}_W Z - \left\langle \overline{\nabla}_W Z, n \right\rangle n) \\
&= [\overline{\nabla}_W V - (\overline{\nabla}_W V)^{\perp}] + [\overline{\nabla}_W Z - (\overline{\nabla}_W Z)^{\perp}] \\
&= (\overline{\nabla}_W V)^T + (\overline{\nabla}_W Z)^T \\
&= \nabla_W V + \nabla_W Z.
\end{aligned}
$$

Thus $\nabla_W(V+Z) = \nabla_W V + \nabla_W Z$.

Finally,

$$\begin{aligned}\langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle &= \left\langle \overline{\nabla}_Z V - (\overline{\nabla}_Z V)^{\perp}, W\right\rangle + \left\langle V, \overline{\nabla}_Z W - (\overline{\nabla}_Z W)^{\perp}\right\rangle \\ &= \langle \overline{\nabla}_Z V, W\rangle + \langle V, \overline{\nabla}_Z W\rangle - \left\langle (\overline{\nabla}_Z V)^{\perp}, W\right\rangle - \left\langle V, (\overline{\nabla}_Z W)^{\perp}\right\rangle \\ &= Z\langle V, W\rangle\end{aligned}$$

since $\left\langle (\overline{\nabla}_Z V)^{\perp}, W\right\rangle = \left\langle V, (\overline{\nabla}_Z W)^{\perp}\right\rangle = 0$ as V and W belong to the tangent vector field on M. Thus $\langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle = Z\langle V, W\rangle$.

Exercise 56. Verify the following sentence:

$$\overline{\nabla}_W V = \nabla_W V + \langle V, S(W)\rangle\, n.$$

Solution. Since $V \in T_pM$ and n is a local Gauss map, it follows that $\langle V, n(p)\rangle = 0$. Thus $W\langle V, n(p)\rangle = 0$. But $W\langle V, n(p)\rangle = \left\langle \overline{\nabla}_W V, n(p)\right\rangle + \left\langle V, \overline{\nabla}_W n(p)\right\rangle$ by Exercise 4 of Lecture Notes 13 solved above. Therefore, $\left\langle \overline{\nabla}_W V, n\right\rangle = -\left\langle V, \overline{\nabla}_W n\right\rangle$. This implies that

$$\begin{aligned}\left\langle \overline{\nabla}_W V, n\right\rangle &= -\left\langle V, \overline{\nabla}_W n\right\rangle \\ &= -\langle V, dn(W)\rangle \\ &= \langle V, -dn(W)\rangle \\ &= \langle V, S(W)\rangle .\end{aligned}$$

Thus

$$\begin{aligned}(\overline{\nabla}_W V)^{\perp} &= \left\langle \overline{\nabla}_W V, n(p)\right\rangle n(p) \\ &= \langle V, S(W_p)\rangle\, n(p) \qquad \text{since } \left\langle \overline{\nabla}_W V, n\right\rangle n = \langle V, S(W)\rangle\end{aligned}$$

which in turn yields

$$\begin{aligned}\overline{\nabla}_W V &= (\overline{\nabla}_W V)^{T} + (\overline{\nabla}_W V)^{\perp} \\ &= \nabla_W V + \langle V, S(W)\rangle\, n.\end{aligned}$$

Therefore $\overline{\nabla}_W V = \nabla_W V + \langle V, S(W)\rangle\, n$ as required.

Exercise 57. Verify the following sentence:

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W]).$$

Solution (1). We shall show that

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W]).$$

Since $\overline{R} \equiv 0$, so the normal component of $\overline{R}(W, W)Z$ equals to 0. This implies that

$$\langle \nabla_V S(W), Z\rangle - \langle \nabla_W S(V), Z)\rangle = \langle S([V, W]), Z\rangle = 0.$$

This is equivalent to

$$\langle \nabla_V S(W) - \nabla_W S(V) - S([V, W]), Z\rangle = 0.$$

In particular, $Z = \nabla_V S(W) - \nabla_W S(V) - S([V, W])$, then

$$\langle \nabla_V S(W) - \nabla_W S(V) - S([V, W]), \nabla_V S(W) - \nabla_W S(V) - S([V, W])\rangle = 0.$$

This implies that

$$\|\nabla_V S(W) - \nabla_W S(V) - S([V, W])\| = 0.$$

Therefore,

$$\nabla_V S(W) - \nabla_W S(V) - S([V, W]) = 0.$$

Thus

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W]).$$

Solution (2). We shall verify that in local coordinates, the equations

$$R(V, W)Z = \langle S(W), Z\rangle\, S(V) - \langle S(V), Z\rangle\, S(W)$$

and

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W]).$$

take on the forms which we had derived earlier.

- The Gauss equations:

In local coordinates, we calculate

$$\begin{aligned} R(X_i, X_j)X_k &= \langle S(X_j), X_k \rangle\, S(X_i) - \langle S(X_i), X_k \rangle\, S(X_j) \\ &= -l_{kj}N_i + l_{ki}N_j, \end{aligned} \tag{1}$$

since $l_{kj} = \langle S(X_j), X_k \rangle$ and $S(X_i) = N_i$

$$\begin{aligned} \nabla_{X_j} X_k &= (\overline{\nabla}_{X_j} X_k)^T \\ &= (X_j X_k)^T \\ &= (X_{kj})^T \\ &= \sum_{i=1}^{2} \Gamma^i_{kj} X_i, \end{aligned}$$

$$\overline{\nabla}_{X_j} X_k = X_{kj},$$

$$\begin{aligned} \left\langle \overline{\nabla}_{X_i} X_j, X_k \right\rangle &= \langle X_{ij}, X_k \rangle \\ &= \sum_{l=1}^{2} \Gamma^l_{ij} \langle X_l, X_k \rangle \\ &= \sum_{l=1}^{2} \Gamma^l_{ij} g_{kl} \end{aligned}$$

since $\langle X_l, X_k \rangle = g_{kl} = g_{lk}$. Using the calculations above we have

$$\begin{aligned} R(X_i, X_j)X_k &= \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i.X_j]}X_k \\ &= \nabla_{X_i}(\sum_{l=1}^{2}\Gamma^l_{kj}X_l) - \nabla_{X_j}(\sum_{l=1}^{2}\Gamma^l_{ki}X_l) && \text{since } [X_i, X_j] = 0, \text{ so } \nabla_{[X_i.X_j]}X_k = 0 \\ &= \sum_{l=1}^{2}[(X_i\Gamma^l_{kj})X_l + \Gamma^l_{kj}\nabla_{X_i}X^l] - \sum_{l=1}^{2}[(X_j\Gamma^l_{ki})X_l + \Gamma^l_{ki}\nabla_{X_j}X_l] \\ &= \sum_{l=1}^{2}[(\Gamma^l_{kj})_i - (\Gamma^l_{ki})_j + \sum_{r=1}^{2}(\Gamma^r_{kj}\Gamma^l_{ir} - \Gamma^r_{ik}\Gamma^l_{jr})]X_l. \end{aligned} \tag{2}$$

Since $N_i = -l^1_i X_1 - l^2_i X_2$, $N_j = -l^1_j X_1 - l^2_j X_2$, so

$$\begin{aligned}-l_{kj}N_i + l_{ki}N_j &= -l_{kj}(-l_i^1 X_1 - l_i^2 X_2) + l_{ki}(-l_j^2 X_1 - l_j^2 X_2).\\ &= (l_{kj}l_i^1 - l_{ki}l_j^1)X_1 + (l_{kj}l_i^2 - l_{ki}l_j^2)X_2.\end{aligned}$$

Combine with (1) we have

$$R(X_i, X_j)X_k = (l_{kj}l_i^1 - l_{ki}l_j^1)X_1 + (l_{kj}l_i^2 - l_{ki}l_j^2)X_2. \tag{3}$$

Since X_1 and X_2 are linearly independent, so comparing coefficients of X_1 and X_2 in (2) and (3) to obtain the classical Gauss equation:

$$(\Gamma_{kj}^r)_i - (\Gamma_{ki}^r)_j + \sum_{l=1}^{2}(\Gamma_{jk}^l\Gamma_{il}^r - \Gamma_{ji}^l\Gamma_{kl}^r) = l_{kj}l_i^r - l_{ki}l_j^r.$$

• The Codazzi-Mainardi equations:
In local coordinates, we have that

$$\begin{aligned}X\langle S(Y), Z\rangle - Y\langle S(X), Z\rangle &= \langle \nabla_X S(Y), Z\rangle + \langle S(Y), \nabla_X Z\rangle - \langle \nabla_Y S(X), Z\rangle - \langle S(X), \nabla_Y Z\rangle\\ &= \langle S(Y), \nabla_X Z\rangle - \langle S(X), \nabla_Y Z\rangle + \langle \nabla_X S(Y) - \nabla_Y S(X), Z\rangle\\ &= \langle S(Y), \nabla_X Z\rangle - \langle S(X), \nabla_Y Z\rangle + \langle S[X,Y], Z\rangle\end{aligned}$$

since $\nabla_X S(Y) - \nabla_Y S(X) = S[X,Y]$. Thus

$$X\langle S(Y), Z\rangle - Y\langle S(X), Z\rangle = \langle S(Y), \nabla_X Z\rangle - \langle S(X), \nabla_Y Z\rangle + \langle S[X,Y], Z\rangle.$$

Let $X = X_k$, $Y = X_j$, $Z = X_i$. Then the above equation becomes

$$X_k\langle S(X_j), X_i\rangle - X_j\langle S(X_k), X_i\rangle = \langle S(X_j), \nabla_{X_k} X_i\rangle - \left\langle S(X_k), \nabla_{X_j} X_i\right\rangle + \langle S[X_k, X_j], X_i\rangle.$$

Hence

$$\begin{aligned}(l_{ij})_k - (l_{ik})_j &= \left\langle S(X_j), \sum_{l=1}^{2}\Gamma_{ki}^l X_l\right\rangle - \left\langle S(X_k), \sum_{l=1}^{2}\Gamma_{ji}^l X_l\right\rangle\\ &= \sum_{l=1}^{2}(\Gamma_{ki}^l l_{jl} - \Gamma_{ji}^l l_{kl}).\end{aligned} \tag{Codazzi-Mainardi}$$

Therefore the classical Codazzi-Mainardi equations have been proved.

EXERCISE 58. Show that if V and W are general vectorfields (not necessarily orthonormal), then

$$K = \frac{R(V, W, W, V)}{\|V \times W\|}.$$

SOLUTION. Firstly, we shall prove that if V and W are orthonormal in the tangent vectofield on M, then $\langle R(V, W)W.V\rangle = K$, where K is the Gaussian curvature. Indeed, since V and W belong to $T_p(M)$, so taking $\{V, W\}$ be a orthonormal basis of T_pM. Since $S\colon T_p(M) \to T_M(M)$, so we can assume $S(V) = aV + bW$ and $S(W) = cV + dW$. We have that

$$\begin{aligned}
&\langle S(V), V\rangle \langle S(W), W\rangle - \langle S(W), V\rangle \langle S(V), W\rangle \\
&= \langle aV + bW, V\rangle \langle cV + dW, W\rangle - \langle cV + dW, V\rangle \langle aV + bW, W\rangle \\
&= ad - bc && \text{since } \langle V, V\rangle = 1, \langle W, W\rangle = 1, \\
& && \langle V, W\rangle = \langle W, V\rangle = 0 \\
&= \det(S) \\
&= K.
\end{aligned}$$

But note that by Gauss's equation

$$\langle R(V, W)W, V\rangle = \langle S(V), V\rangle \langle S(W), W\rangle - \langle S(W), V\rangle \langle S(V), W\rangle .$$

Therefore

$$\langle R(V, W)W, V\rangle = K.$$

Now, we consider the case that V and W are general vectorfields (not necessarily orthonormal).
Let

$$V' = \frac{V}{\|V\|} \qquad \text{and} \qquad W' = x\frac{V}{\|V\|} + yW \qquad \text{where } x, y \in \mathbf{R}.$$

We find x and y such that V' and W' are orthonormal. This is equivalent to

$$\left\langle W', V'\right\rangle = 0, \tag{1}$$
$$\left\langle W', W'\right\rangle = 1. \tag{2}$$

(Note that $\langle V', V'\rangle = 1$). We have that

$$\langle W', V' \rangle = \left\langle x\frac{V}{\|V\|} + yW, \frac{V}{\|V\|} \right\rangle = x + y\frac{\langle W, V \rangle}{\|V\|},$$
$$\langle W', W' \rangle = \left\langle x\frac{V}{\|V\|} + yW, x\frac{V}{\|V\|} + yW \right\rangle = x^2 + 2xy\frac{\langle W, V \rangle}{\|V\|} + y^2 \|W\|^2 .$$

Thus (1) and (2), respectively, become

$$x + y\frac{\langle W, V \rangle}{\|V\|} = 0, \qquad (3)$$
$$x^2 + 2xy\frac{\langle W, V \rangle}{\|V\|} + y^2 \|W\|^2 = 1. \qquad (4)$$

It follows from (3) that $x = -y\frac{\langle W,V \rangle}{\|V\|}$ and substitute it into (4) to obtain

$$y^2\frac{\langle W, V \rangle^2}{\|V\|^2} - 2y^2\frac{\langle W, V \rangle^2}{\|V\|^2} + y^2 \|W\|^2 = 1$$

or

$$y^2\left(\frac{\|V\|^2 \|W\|^2 - \langle W, V \rangle^2}{\|V\|^2}\right) = 1.$$

Thus

$$y^2\frac{\|V \times W\|^2}{\|V\|^2} = 1 \qquad \text{by Lagrange's identity.}$$

Therefore $y = \frac{\|V\|}{\|V \times W\|}$ and $x = -\frac{\langle W,V \rangle}{\|V \times W\|}$. Now vectors

$$V' = \frac{V}{\|V\|} \qquad \text{and} \qquad W' = \frac{\langle W, V \rangle}{\|V \times W\|} \cdot \frac{V}{\|V\|} + \frac{\|V\|}{\|V \times W\|} \cdot W$$

are orthonormal. Apply the above result for V' and $W'$to obtain

$$
\begin{aligned}
K &= \langle R(V', W')W', V'\rangle \\
&= \left\langle R\left(\frac{V}{\|V\|}, x\frac{V}{\|V\|} + yW\right)\left(x\frac{V}{\|V\|} + yW\right), \frac{V}{\|V\|}\right\rangle \\
&= \frac{1}{\|V\|^2}\left\langle R\left(V, x\frac{V}{\|V\|} + yW\right)\left(x\frac{V}{\|V\|} + yW\right), V\right\rangle && \text{since } R \text{ is linear in every variable} \\
&= \frac{y^2}{\|V\|^2}\langle R(V,W)W, V\rangle && \begin{array}{c}\text{since } \langle R(V,W)V,V\rangle = 0, \\ \langle R(V,V)V,V\rangle = 0, \langle R(V,V)W,V\rangle = 0\end{array} \\
&= \frac{\|V\|^2}{\|V\times W\|^2\,\|V\|^2}\langle R(V,W)W, V\rangle \\
&= \frac{R(V,W,W,V)}{\|V\times W\|^2},
\end{aligned}
$$

as required.

Exercise 59. Show that the absolute geodesic curvature of great circles in a sphere and helices on a cylinder are everywhere zero.

Solution. Firstly, by choosing a suitable coordinates system, we can assume that the great circle in a sphere is

$$[0, 2\pi] \ni t \xrightarrow{\alpha} \alpha(t) = (r\cos t, r\sin t, 0).$$

We have that

$$
\begin{aligned}
\alpha'(t) &= (-R\sin t, R\cos t, 0), \\
\alpha''(t) &= (-R\cos t, -R\sin t, 0).
\end{aligned}
$$

Since $n(p) = \frac{1}{R}p$ is a Gauss map in a neighborhood of $\alpha(t)$, it follows that

$$
\begin{aligned}
(\alpha''(t))^T &= \alpha''(t) - \left\langle \alpha''(t), (\cos t, \sin t, 0)\right\rangle (\cos t, \sin t, 0) \\
&= (-R\cos t, -R\sin t, 0) + (R\cos t, R\sin t, 0) \\
&= (0, 0, 0).
\end{aligned}
$$

Thus

$$|\kappa_g| = \left\|(\alpha''(t))^T\right\| = 0.$$

Therefore, the absolute geodesic curvature of great circles in a sphere is everywhere zero.

The equation of the helix is

$$\mathbf{R} \ni t \overset{\alpha}{\longmapsto} \alpha(t) = (a\cos t, a\sin t, bt).$$

We have that

$$\begin{aligned}
\alpha'(t) &= (-a\sin t, a\cos t, 0),\\
\alpha''(t) &= (-a\cos t, -a\sin t, 0).
\end{aligned}$$

The equation of the cyclinder surface is

$$X(u,v) = (a\cos u, a\sin u, v).$$

We have that

$$\begin{aligned}
X_1 &= (-a\sin u, a\cos u, 0),\\
X_2 &= (0,0,1),\\
N(u,v) &= \frac{X_1 \times X_2}{\|X_1 \times X_2\|} = \frac{1}{a}(a\cos u, a\sin u, 0)\\
&= (\cos u, \sin u, 0),\\
n(p) = N \circ X^{-1}(p) &= (\frac{p_1}{a}, \frac{p_2}{a}, 0), \qquad p = (p_1, p_2, p_3).
\end{aligned}$$

This implies that

$$n_p(t) = (\cos t, \sin t, 0) \qquad \begin{gathered}\text{where } n_p(t) = n(p(t)),\\ p(t) = (a\cos t, a\sin t, bt).\end{gathered}$$

Thus

$$\begin{aligned}
(\alpha''(t))^T &= \alpha''(t) - \left\langle \alpha''(t), n_p(t) \right\rangle n_p(t)\\
&= (-a\cos t, -a\sin t, 0) + a(\cos t, \sin t, 0)\\
&= (0,0,0).
\end{aligned}$$

Hence

$$|\kappa_g| = \left\|(\alpha''(t))^T\right\| = 0.$$

Therefore, the absolute geodesic curvature of helices on a cylinder are everywhere zero.

Exercise 60. Let $\mathbf{S}^2$ be oriented by its outward unit normal, i.e., $n(p) = p$, and compute the geodesic curvature of the circles in $\mathbf{S}^2$ which lie in planes $z = h$, $-1 < h < 1$. Assume that all these circles are oriented consistenly with respect to the rotation about the z-axis.

Solution (1). The equation for the circle in $\mathbf{S}^2$ which lie in planes $z = h$, $-1 < h < 1$ is

$$\alpha(t) = (\sqrt{1-h^2}\cos t, \sqrt{1-h^2}\sin t, h).$$

Let $r = \sqrt{1-h^2}$ to obtain

$$\alpha(t) = (r\cos t, r\sin t, h).$$

Thus

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t, 0),\\
\alpha''(t) &= (-r\cos t, -r\sin t, 0),\\
n(p) &= p.
\end{aligned}$$

This implies that

$$n(p(t)) = (r\cos t, r\sin t, h).$$

Hence

$$J\alpha' = n \times \alpha' = (-hr\cos t, -hr\sin t, r^2).$$

Therefore

$$\begin{aligned}
\kappa_g &= \frac{\langle \alpha'', J\alpha' \rangle}{\|\alpha'\|^3} \\
&= \frac{hr^2\cos^2 t + hr^2\sin^2 t}{r^3} \\
&= \frac{hr^2(\cos^2 t + \sin^2 t)}{r^3} \\
&= \frac{hr^2}{r^3} & \text{since } \cos^2 t + \sin^2 t = 1 \\
&= \frac{h}{r} \\
&= \frac{h}{\sqrt{1-h^2}}.
\end{aligned}$$

SOLUTION (2). The equation parametrized by arclength for the circle in $\mathbf{S}^2$ which lie in planes $z = h$, $-1 < h < 1$ is

$$\alpha(t) = (\sqrt{1-h^2}\cos\frac{t}{\sqrt{1-h^2}}, \sqrt{1-h^2}\sin\frac{t}{\sqrt{1-h^2}}, h).$$

Let $r = \sqrt{1-h^2}$ to obtain

$$\alpha(t) = (r\cos\frac{t}{r}, r\sin\frac{t}{r}, h).$$

Thus

$$\begin{aligned}
\alpha'(t) &= (-\sin\frac{t}{r}, \cos\frac{t}{r}, 0), \\
\alpha''(t) &= (-\frac{1}{r}\cos\frac{t}{r}, -\frac{1}{r}\sin\frac{t}{r}, 0), \\
n(p) &= p.
\end{aligned}$$

This implies that

$$n(p(t)) = (r\cos\frac{t}{r}, r\sin\frac{t}{r}, h).$$

Hence

$$J\alpha' = n \times \alpha' = (-h\cos\frac{t}{r}, -h\sin\frac{t}{r}, r).$$

Therefore

$$\begin{aligned}
\kappa_g &= \left\langle \alpha'', J\alpha' \right\rangle \\
&= \frac{h}{r}\cos^2\frac{t}{r} + \frac{h}{r}\sin^2\frac{t}{r} \\
&= \frac{h}{r}(\cos^2\frac{t}{r} + \sin^2\frac{t}{r}) \\
&= \frac{h}{r} && \text{since } \cos^2\frac{t}{r} + \sin^2\frac{t}{r} = 1 \\
&= \frac{h}{\sqrt{1-h^2}}.
\end{aligned}$$

EXERCISE 61. Show that if α is a geodesic, then it must have constant speed.

SOLUTION. Define $\widetilde{\alpha}' \colon \alpha(I) \to \mathbf{R}^3$ given by $\widetilde{\alpha}'(\alpha(t)) = \alpha'(t)$. We have that

$$\begin{aligned}
\overline{\nabla}_{\alpha'(t)}\widetilde{\alpha}' &= (\alpha'(t)\alpha_1'(t), \alpha'(t)\alpha_2'(t), \alpha'(t)\alpha_3'(t)) \\
&= (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t)) \\
&= \alpha''(t),
\end{aligned}$$

where $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$. Thus

$$\nabla_{\alpha'(t)}\widetilde{\alpha}' = \alpha''(t).$$

Hence

$$\begin{aligned}
\left\langle \alpha'', J\alpha' \right\rangle &= \left\langle (\alpha'')^T, J\alpha' \right\rangle \\
&= \left\langle \nabla_{\alpha'}\widetilde{\alpha}', J\alpha' \right\rangle.
\end{aligned}$$

This implies that

$$\kappa_g = \frac{\langle \nabla_{\alpha'}\widetilde{\alpha}', J\alpha \rangle}{\|\alpha'\|^3}.$$

Since α is a geodesic, then $\kappa_g = 0$, so $\langle \nabla_{\alpha'}\widetilde{\alpha}', J\alpha' \rangle = 0$. Since $J\alpha' \in T_pM$ so from $\langle \nabla_{\alpha'}\widetilde{\alpha}', J\alpha' \rangle = 0$, we deduce that $\nabla_{\alpha'}\widetilde{\alpha}'$ belongs to the orthogonal complement of T_pM, but by the definition of $\nabla_{\alpha'}\widetilde{\alpha}'$, we have that $\nabla_{\alpha'}\widetilde{\alpha}' \in T_p(M)$. Therefore $\nabla_{\alpha'}\widetilde{\alpha}' \in T_p(M) \cap (T_pM)^{\perp} = \{0\}$. This implies that $\nabla_{\alpha'}\widetilde{\alpha}' = 0$. Thus $\overline{\nabla}_{\alpha'}\widetilde{\alpha}' = (\overline{\nabla}_{\alpha'}\widetilde{\alpha}')^{\perp}$. Therefore

$$\begin{aligned}
\left\langle \alpha', \alpha' \right\rangle' &= \left\langle \alpha'', \alpha' \right\rangle + \left\langle \alpha', \alpha'' \right\rangle \\
&= 2\left\langle \alpha', \alpha'' \right\rangle \\
&= 2\left\langle \overline{\nabla}_{\alpha'}\widetilde{\alpha}', \alpha' \right\rangle \\
&= 2\left\langle (\overline{\nabla}_{\alpha'}\widetilde{\alpha}')^{\perp}, \alpha' \right\rangle \\
&= 0,
\end{aligned}$$

giving that $\|\alpha'\| = \text{const}$.

Exercise 62. Show that if $M = \mathbf{R}^2$, and $n = (0, 0, 1)$, then J is clockwise rotation about the origin by $\pi/2$.

Solution. Since $M = \mathbf{R}^2$, we have $T_pM = M = \mathbf{R}^2$. For $n = (0, 0, 1)$, any $V \in T_pM$ is of the form $V = (V^1, V^2, 0)$. We have that

$$\begin{aligned}
JV &= n \times V \\
&= (\begin{vmatrix} 0 & 1 \\ V^2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & V^1 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ V^1 & V^2 \end{vmatrix}) \\
&= (-V^2, V^1, 0)
\end{aligned}$$

and

$$\langle V, JV \rangle = -V^1V^2 + V^2V^1 = 0.$$

This implies that $V \perp JV$. Let

$$\begin{aligned}
W &= V + JV \\
&= (V^1 - V^2, V^1 + V^2, 0).
\end{aligned}$$

We have that

$$\begin{aligned}\langle W, V\rangle &= V^1(V^1 - V^2) + V^2(V^1 + V^2)\\ &= (V^1)^2 + (V^2)^2 \geq 0.\end{aligned}$$

Thus

$$(V, JV) = -\frac{\pi}{2}.$$

It follows from the facts that $V \perp JV$ and $(V, JV) = -\frac{\pi}{2}$ proved above that J is the clockwise rotation about the origin by $\pi = \pi/2$.

EXERCISE 63. Verify the following two equations:

$$\overline{\alpha}'' = \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\left\langle \alpha''(s^{-1}), \alpha'(s^{-1})\right\rangle}{\|\alpha'(s^{-1})\|^4},$$

$$\kappa_g = \frac{\langle \alpha'', Ja'\rangle}{\|\alpha'\|^3}.$$

SOLUTION. Firstly, we shall verify that

$$\overline{\alpha}'' = \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\left\langle \alpha''(s^{-1}), \alpha'(s^{-1})\right\rangle}{\|\alpha'(s^{-1})\|^4}.$$

Since $(s^{-1})' = \frac{1}{\|\alpha'\|}$, be the chain rule

$$\begin{aligned}\alpha'(t) &= \alpha'(s^{-1}(t))[s^{-1}(t)]'\\ &= \alpha'(s^{-1}(t))\frac{1}{\|\alpha'(s^{-1}(t))\|}.\end{aligned}$$

Differentiate both sides of the above equation to obtain

$$\begin{aligned}\overline{\alpha}'' &= \alpha''(s^{-1})(s^{-1})' \cdot \frac{1}{\|\alpha'(s^{-1})\|} + [\frac{1}{\|\alpha'(s^{-1})\|}]' \cdot \alpha'(s^{-1}).\\ &= \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1})[\frac{1}{\|\alpha'(s^{-1})\|}]'. \qquad (1)\end{aligned}$$

Since

$$\left\|\alpha'(s^{-1})\right\|^2 = \left\langle \alpha'(s^{-1}), \alpha'(s^{-1}) \right\rangle,$$

so

$$(\left\|\alpha'(s^{-1})\right\|^2)' = 2\left\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \right\rangle (s^{-1})'.$$

Thus

$$2\left\|\alpha'(s^{-1})\right\| \left\|\alpha'(s^{-1})\right\|' = \frac{2\left\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \right\rangle}{\|\alpha'(s^{-1})\|}.$$

Hence

$$\left\|\alpha'(s^{-1})\right\|' = \frac{\left\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \right\rangle}{\|\alpha'(s^{-1})\|^2},$$

and so

$$\begin{aligned}(\frac{1}{\|\alpha'(s^{-1})\|})' &= (\left\|\alpha'(s^{-1})\right\|^{-1})' \\ &= -\left\|\alpha'(s^{-1})\right\|^{-2}\left\|\alpha'(s^{-1})\right\|' \\ &= -\frac{\left\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \right\rangle}{\|\alpha'(s^{-1})\|^4}. \end{aligned} \tag{2}$$

Substitute (2) into (1) to obtain

$$\overline{\alpha}'' = \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\left\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \right\rangle}{\|\alpha'(s^{-1})\|^4},$$

as required.

Secondly, we shall verify that

$$\kappa_g = \frac{\langle \alpha'', Ja' \rangle}{\|\alpha'\|^3}.$$

Indeed,

$$
\begin{aligned}
&\kappa_g \\
&= \overline{\kappa}_g \\
&= \langle \overline{\alpha}'', J\overline{a}' \rangle \\
&= \left\langle \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\langle \alpha''(s^{-1}), \alpha'(s^{-1}) \rangle}{\|\alpha'(s^{-1})\|^4}, J\overline{a}' \right\rangle \\
&= \left\langle \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2}, \frac{1}{\|\alpha'(s^{-1})\|} J\overline{a}' \right\rangle && \text{since} \quad \begin{array}{c} \langle \alpha'(s^{-1}), J\overline{a}'(s^{-1}) \rangle \\ = \quad \frac{1}{\|\alpha'\|} \langle \alpha', J\alpha' \rangle = 0 \end{array} \\
&= \frac{\langle \alpha'', Ja' \rangle}{\|\alpha'\|^3}, && \text{since } J(tV) = tJV, t \in \mathbf{R},
\end{aligned}
$$

as required.

EXERCISE 64. Show that if α is parametrized by arclength, then

$$|\kappa_g| = \left\| \nabla_{\alpha'} \widetilde{\alpha}' \right\|.$$

SOLUTION. It is clear that

$$\overline{\nabla}_{\alpha'(t)} \widetilde{\alpha}' = \alpha''(t)$$

since $\overline{\nabla}_{\alpha'(t)} \widetilde{\alpha}' = [\widetilde{\alpha}' \circ \alpha(t)]' = [\alpha'(t)]' = \alpha''(t)$. Since α is parametrized by arclength, it follows that

$$
\begin{aligned}
|\kappa_g| &= \left\| (\alpha'')^T \right\| \\
&= \left\| (\overline{\nabla}_{\alpha'(t)} \widetilde{\alpha}')^T \right\| \\
&= \left\| \nabla_{\alpha'(t)} \widetilde{\alpha}' \right\| \\
&= \left\| \nabla_{\alpha'} \widetilde{\alpha}' \right\|,
\end{aligned}
$$

as required.

EXERCISE 65. Show that is α a geodesic if and only if $\nabla_{\alpha'} \widetilde{\alpha}' \equiv 0$.

SOLUTION. Suppose first that $\nabla_{\alpha'} \widetilde{\alpha}' \equiv 0$. Then

$$\kappa_g = \frac{\langle \nabla_{\alpha'} \widetilde{\alpha}', J\alpha' \rangle}{\|\alpha'\|^3} = 0.$$

Thus α is a geodesic.

Conversely, if α a geodesic, i.e., $\kappa_g = 0$, then $\overline{\alpha} = \alpha \circ s^{-1}$ is a parametrization by arclength, so by Exercise 6 of Lecture Notes 15 proved above, we have that

$$
\begin{array}{rrl}
 & \nabla_{\overline{\alpha}'}\widetilde{\overline{\alpha}}' = 0 & \\
\Longleftrightarrow & \nabla_{(\alpha\circ s^{-1})'}\widetilde{\overline{\alpha}}' = 0 & \\
\Longleftrightarrow & \nabla_{\alpha'(s^{-1})(s^{-1})'}\widetilde{\overline{\alpha}}' = 0 & \\
\Longleftrightarrow & (s^{-1})'\nabla_{\alpha'}\widetilde{\overline{\alpha}}' = 0 & \text{since } \nabla_{fU}V = f\nabla_U V \\
\Longrightarrow & \nabla_{\alpha'}[\alpha\circ(s^{-1})]' = 0 & \text{since } (s^{-1})' = \dfrac{1}{\|\alpha'\|} \neq 0 \\
\Longrightarrow & \nabla_{\alpha'}\alpha'(s^{-1})(s^{-1})' = 0 & \\
\Longrightarrow & \alpha'(s^{-1})'\alpha' + (s^{-1})'\nabla_{\alpha'}\widetilde{\alpha}' = 0 & \text{use } \nabla_U fV = UfV + f\nabla_U V \\
\Longrightarrow & \nabla_{\alpha'}\widetilde{\alpha}' = 0 & \begin{array}{c}\text{since } (s^{-1})' = \text{const because } \alpha \text{ is a geodesic,} \\ \text{and so has constant speed, and thus } \alpha'(s^{-1})' = 0.\end{array}
\end{array}
$$

Therefore, α a geodesic if and only if $\nabla_{\alpha'}\widetilde{\alpha}' \equiv 0$.

Exercise 66. Write down the equations of the geodesic in a surface of revolution. In particular, verify that the great circles in a sphere are geodesics.

Solution. We first compute the Christoffel symbols of a surface of revolution. Consider a surface of revolution parametrized by

$$
\begin{gathered}
x(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)) \\
0 < u < 2\pi, \qquad a < v < b, \qquad \psi(v) \neq 0.
\end{gathered}
$$

We have that

$$X_1 = (-\varphi(v)\sin u, \varphi(v)\cos u, 0)$$

$$X_2 = (\varphi'(v)\cos u, \varphi'(v)\sin u, \psi'(v))$$

$$X_{11} = (-\varphi(v)\cos u, -\varphi(v)\sin u, 0)$$

$$X_{12} = (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0)$$

$$X_{21} = (-\varphi'(v)\sin u, \varphi'(v)\cos u, 0)$$

$$X_{22} = (\varphi''(v)\cos u, \varphi''(v)\sin u, \psi''(v))$$

$$g_{11} = \langle X_1, X_1\rangle = \varphi^2(v)\sin^2 u + \varphi^2(v)\cos^2 u = \varphi^2(v)$$

$$g_{12} = \langle X_1, X_2\rangle = 0 = g_{21}$$

$$\begin{aligned} g_{22} &= [\varphi'(v)]^2\cos^2 u + [\varphi'(v)]^2\sin^2 u + [\psi'(v)]^2 \\ &= [\varphi'(v)]^2 + [\psi'(v)]^2 = 1 \end{aligned}$$

since assume that the rotating curve is parametrized by arc-length.Thus

$$\begin{aligned} (g_{ij}) &= \begin{pmatrix} \varphi^2 & 0 \\ 0 & 1 \end{pmatrix}, \\ (g^{ij}) &= (g_{ij})^{-1} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}. \end{aligned}$$

- Γ^1_{11}, Γ^2_{11}: We have that

$$\begin{aligned} &\langle X_{11}, X_1\rangle = 0, \\ &\langle X_{11}, X_2\rangle = -\varphi(v)\varphi'(v) = -\varphi\varphi'. \end{aligned}$$

Thus $\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}\begin{pmatrix} 0 \\ -\varphi\varphi' \end{pmatrix} = \begin{pmatrix} 0 \\ -\varphi\varphi' \end{pmatrix}$. Hence $\Gamma^1_{11} = 0$, $\Gamma^2_{11} = -\varphi\varphi'$.

- Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}: We have that

$$\begin{aligned} &\langle X_{12}, X_1\rangle = \varphi\varphi', \\ &\langle X_{12}, X_2\rangle = 0. \end{aligned}$$

Thus $\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix} 1 & 0 \\ 0 & \varphi^2 \end{pmatrix}\begin{pmatrix} \varphi\varphi' \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\varphi'}{\varphi} \\ 0 \end{pmatrix}$. Hence $\Gamma^1_{12} = \frac{\varphi'}{\varphi} = \Gamma^1_{21}$, $\Gamma^2_{12} = 0 = \Gamma^2_{21}$ (Note that $\Gamma^k_{ij} = \Gamma^k_{ji}$).

- Γ^1_{22}, Γ^2_{22}: We have that

$$\langle X_{22}, X_1\rangle = 0,$$
$$\langle X_{22}, X_2\rangle = \varphi'\varphi'' + \psi'\psi''.$$

Thus $\begin{pmatrix}\Gamma_{22}^1\\ \Gamma_{22}^2\end{pmatrix} = \frac{1}{\varphi^2}\begin{pmatrix}1 & 0\\ 0 & \varphi^2\end{pmatrix}\begin{pmatrix}0\\ \varphi'\varphi'' + \psi'\psi''\end{pmatrix} = \begin{pmatrix}0\\ \varphi'\varphi'' + \psi'\psi''\end{pmatrix}$. Hence $\Gamma_{22}^1 = 0$, $\Gamma_{22}^2 = \varphi'\varphi'' + \psi'\psi''$.

The two equations

$$u_1'' + \Gamma_{11}^1(u_1')^2 + 2\Gamma_{12}^1 u_1'u_2' + \Gamma_{22}^1(u_2')^2 = 0,$$
$$u_2'' + \Gamma_{11}^2(u_1')^2 + 2\Gamma_{12}^2 u_1'u_2' + \Gamma_{22}^2(u_2')^2 = 0,$$

become

$$u_1'' + 2\frac{\varphi'}{\varphi}u_1'u_2' = 0, \tag{1}$$
$$u_2'' - \varphi\varphi'(u_1')^2 + (\varphi'\varphi'' + \psi'\psi'')(u_2')^2 = 0. \tag{2}$$

The equation of the sphere centered at 0 of radius r is

$$x(u, v) = (r\sin v\cos u, r\sin v\sin u, r\cos v).$$

In this case, we have that $\varphi(t) = r\sin t$ and $\psi(t) = r\cos t$, and so the two equations (1) and (2) become

$$u_1'' + 2\frac{\cos t}{\sin t}u_1'u_2' = 0, \tag{3}$$
$$u_2'' - r^2\sin t\cos t(u_1')^2 = 0. \tag{4}$$

Without loss of generality, we assume that the great circle $\alpha(t)$ lies in the xz-plane and so its equation is

$$\alpha(t) = (r\sin t, 0, r\cos t).$$

The function $u(t)$ satifies $x(u(t)) = \alpha(t)$ be $u(t) = (0, t)$, i.e, $u_1(t) = 0$ and $u_2(t) = t$. The functions $u_1(t)$ and $u_2(t)$ satisfy the two equations (3) and (4). Thus the great circles in a sphere are geodesics.

EXERCISE 67. Show that the sum of the angles in a triangle is π.

SOLUTION. Every line in a plane is a geodesic curve and has Gaussian curvature $\kappa = 0$, $\sum_{i=1}^{3} \theta_i = \pi$. By Gauss-Bonnet Theorem,

$$\sum_{i=0}^{2} \int_{s_i}^{s_{i+1}} \kappa_g(S)\, ds + \iint_R \kappa\, d\sigma + \sum_{i=0}^{2} \theta_i = 2\pi.$$

Since $\kappa = 0$, so the equation above becomes

$$\sum_{i=0}^{2} \theta_i = 2\pi.$$

But

$$\sum_{i=0}^{2} \theta_i = 3\pi - \sum_{i=0}^{2} \alpha_i,$$

where α_i, $i = 0, 1, 2$, is the internal angles of the triangle. Thus

$$3\pi - \sum_{i=0}^{2} \alpha_i = 2\pi,$$

and so

$$\sum_{i=0}^{2} \alpha_i = \pi.$$

Therefore the sum of the angles in a triangle is π.

EXERCISE 68. Show that the total geodesic curvature of a simple closed planar curve is 2π.

SOLUTION. Since a simple closed planar curve has $\kappa = 0$ and $\sum_{i=1}^{3} \theta_i = 0$, it follows from Gauss-Bonnet Theorem that

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \kappa_g(S)\, ds = 2\pi.$$

Thus by Corollary 12 of Lecture Notes 15, the total geodesic curvature of a simple closed planar curve is 2π.

EXERCISE 69. Show that the Gaussian curvature of a surface which is homeomorphic to the torus must alwasy be equal to zero at some point.

SOLUTION. Let S be the surface of the torus and M a surface which is homeomorphic to S. Let f be the homeomorphism from M onto S. The Euler-Poincare characteristic of the torus S is $\chi(S) = 0$ and since M is homeomorphic to S, so $\chi(M) = \chi(S) = 0$. Since S is a compact surface and M is homeomorphic to S, so M is also a compact surface (since the image of a compact set under a continuous map is also a compact set). Thus, by Corollary 2 of the Gauss-Bonnet Theorem (see p. 280 of [Car92]), (or since the surface M is homeomorphic to the torus, so its boundary is empty, and so the following expression is followed from the Gauss-Bonnet theorem)

$$\iint_M \kappa \, d\sigma = 2\pi\chi(M) = 0,$$

where κ is the Gaussian curvature over M. Therefore, there exists some point p of M such that $\kappa(p) = 0$ because κ is a continuous function on M, if there are $p_1, p_2 \in M$ such that $\kappa(p_1) > 0$ and $\kappa(p_2) < 0$, then by the intermediate value theorem, there is $q \in M$ satisfying $\kappa(q) = 0$ and it is clear that if either $\kappa > 0$ for all $p \in M$ or $\kappa < 0$ for all $p \in M$ occur, then this contradicts to the fact that $\iint_M \kappa \, d\sigma = 0$.

EXERCISE 70. Show that a simple closed curve with total geodesic curvature zero on a sphere bisects the area of the sphere.

SOLUTION. The parametrization of the sphere of radius r is

$$x(\theta, \varphi) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta).$$

We have that

$$x_\theta = (r \cos\theta \cos\varphi, r \cos\theta \sin\varphi, -r \sin\theta),$$

$$x_\varphi = (-r \sin\theta \sin\varphi, r \sin\theta \cos\varphi, 0),$$

$$x_{\theta\theta} = (-r \sin\theta \cos\varphi, -r \sin\theta \sin\varphi, -r \cos\theta),$$

$$x_{\theta\varphi} = (r \cos\theta \sin\varphi, r \cos\theta \cos\varphi, 0),$$

$$x_{\varphi\varphi} = (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, 0),$$

$$x_\theta \times x_\varphi = (r^2\sin^2\theta\cos\varphi, r^2\sin^2\theta\sin\varphi, r^2\sin\theta\cos\theta),$$

$$\begin{aligned}\|x_\theta \times x_\varphi\| &= \sqrt{r^4\sin^4\theta(\cos^2\varphi+\sin^2\varphi)+r^4\sin^2\theta\cos^2\theta} \\ &= \sqrt{r^4\sin^2\theta(\sin^2\theta+\cos^2\theta)} = r^2\sin\theta,\end{aligned}$$

$$\begin{aligned}n &= \frac{x_\theta \times x_\varphi}{\|x_\theta \times x_\varphi\|} \\ &= (\sin\theta\sin\varphi, \sin\theta\sin\varphi, \cos\theta),\end{aligned}$$

$$E = \langle x_\theta, x_\theta\rangle = r^2,$$

$$F = \langle x_\theta, x_\varphi\rangle = 0,$$

$$G = \langle x_\varphi, x_\varphi\rangle = r^2\sin^2\theta,$$

$$e = \langle n, x_{\theta\theta}\rangle = -R,$$

$$f = \langle n, x_{\theta\varphi}\rangle = 0,$$

$$g = \langle n, x_{\varphi\varphi}\rangle = -r\sin^2\varphi,$$

$$\begin{aligned}\kappa &= \frac{eg-f^2}{EG-F^2} \\ &= \frac{r^2\sin^2\theta}{r^4\sin^2\theta} \\ &= \frac{1}{r^2}.\end{aligned}$$

By Gauss-Bonnet Theorem,

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\kappa_g(S)\,ds+\iint_R\kappa\,d\sigma+\sum_{i=0}=2\pi$$

where R is the rehion bounded by the simple closed curve. Since the curve is simple closed, it follows that $\sum_{i=0}^{k}\theta_i=0$. We also have that $\kappa=\frac{1}{R^2}$ and $\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\kappa_g(s)\,ds=0$ because the total geodesic curvature of the curve is zero, and so the above equation becomes

$$\iint_R\frac{1}{r^2}\,d\sigma=2\pi.$$

Thus $\frac{1}{r^2}\mathrm{Area}(R)=2\pi$ since $\iint_R d\sigma=\mathrm{Area}(R)$ where $\mathrm{Area}(R)$ is the area of the region R. Hence $\mathrm{Area}(R)=2\pi r^2=\frac{1}{2}$(Area of the sphere surface).

Exercise 71. Show that there exists at most one closed geodesic on a cylinder with negative curvature.

Solution. If there are two geodesics γ_1 and γ_2 which start from $p\in S$ (the cylinder) and they meet at again at a point $q\in S$ in such a way that the traces of γ_1 and γ_2 constitute the boundary of a simple region R of S, then by Gauss-Bonnet Theorem $\iint_R\kappa\,d\sigma+\theta_1+\theta_2=2\pi$ where θ_1 and θ_2 are the external angles of the region R. Since geodesics γ_2 and γ_2 cannot be mutually tangent. we have $\theta_i<\pi$, $i=1,2$. On the other hand, $\kappa<0$, whence the contradiction.

Thus if S contains one closed geodesic Γ and another closed geodesic Γ', then Γ and Γ' does not intersect Γ'. Otherwise, the arcs of Γ and Γ' between two consecutive intersectionpoints r_1 and r_2, would be the boundary of a simple region, contradicting above.

Applying the Gauss-Bonnet Theorem to the region R bounded by two simple non-intersecting geodesics Γ and Γ' of S to obtain

$$\iint_R\kappa\,d\sigma=2\pi\chi(R)=0\qquad\qquad\text{since }\chi(R)=0$$

which is a contradictiobn, since $\kappa<0$.

Therefore there exists at most one closed geodesic on a cylinder with negative curvature.

Exercise 72. Show that the area of a geodesic polygon with k vertices on a sphere of radius 1 is equal to the sum of its angles minus $(k-2)\pi$.

SOLUTION. We have that $F = k-2$, $E = k$, $V = k$. Thus $\chi = F - E + V = k - 2$. By Gauss-Bonnet Theorem,

$$\sum_{i=1}^{k} \int_{c_i} \kappa_g(s)\, ds + \iint_R \kappa\, d\sigma + \sum_{i=1}^{k} \theta_i = 2(k-2)\pi.$$

But since $\kappa = 1$, $\sum_{i=1}^{k} \int_{c_i} \kappa_g(s)\, ds = 0$ (since the sides of a geodesic polygon are geodesics), $\iint_R 1\, d\sigma = \text{Area}(R)$, $\sum_{i=1}^{k} \theta_i = 3\pi(k-2) - \sum_{i=1}^{k} \alpha_i$ (since a geodesic polygon is divided into $k-2$ triangles), where α_i are the internal angles of a geodesic polygon so the equation above becomes

$$\begin{aligned} \text{Area}(R) &= \sum_{i=1}^{k} \alpha_i - 3(k-2)\pi + 2(k-2)\pi \\ &= \sum_{i=1}^{k} \alpha_i - (k-2)\pi. \end{aligned}$$

EXERCISE 73. Let p be a point of a surface M, T be a geodesic triangle which contains p, and α, β, γ be the angles of T. Show that

$$\kappa(p) = \lim_{T \to p} \kappa(\xi) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{\text{Area}(T)}.$$

In particular, note that the above proves Gauss's Theorema Egregium.

SOLUTION. Since T is a geodesic triangle, so $\sum_{i=1}^{3} \int_{C_i} \kappa_g(S)\, ds = 0$ and $F = 1$, $V = E = 3$. Thus $\chi = F - E + V = 1$. Further, $\sum_{i=1}^{3} \theta_i = 3\pi - (\alpha + \beta + \gamma)$, where θ_i, $i = 1, 2, 3$, are the external angles of the triangle T. Applying the Gauss-Bonnet Theorem to obtain

$$\iint_T \kappa\, d\sigma + 3\pi - (\alpha + \beta + \gamma) = 2\pi.$$

Thus

$$\iint_T \kappa\, d\sigma = \alpha + \beta + \gamma - \pi. \tag{1}$$

By the mean value theorem,

$$\iint_T \kappa\, d\sigma = \kappa(\xi)\text{Area}(T)$$

where ξ is some point in T. Therefore, the equation (1) becomes

$$\kappa(\xi) = \frac{\alpha + \beta + \gamma - \pi}{\text{Area}(T)}.$$

Hence

$$\kappa(p) = \lim_{T \to p} \kappa(\xi) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{\text{Area}(T)},$$

as required.

Exercise 74. Show that the sum of the angles of a geodesic triangle on a surface of positive curvature is more than π, and on a surface of negative curvature is less than π.

Solution. Since T is a geodesic triangle, so $\sum_{i=1}^{3} \int_{C_i} \kappa_g(S)\, ds = 0$ and $F = 1$, $V = E = 3$. Thus $\chi = F - E + V = 1$. Further, $\sum_{i=1}^{3} \theta_i = 3\pi - (\alpha + \beta + \gamma)$, where θ_i, $i = 1, 2, 3$, are the external angles of the triangle T. Applying the Gauss-Bonnet Theorem to obtain

$$\iint_T \kappa\, d\sigma + 3\pi - (\alpha + \beta + \gamma) = 2\pi.$$

Thus

$$\iint_T \kappa\, d\sigma = \alpha + \beta + \gamma - \pi. \qquad (1)$$

If $\kappa > 0$, then $\iint_T \kappa\, d\sigma > 0$ and so we deduce from (1) that $\alpha + \beta + \gamma > \pi$. If $\kappa < 0$, then $\iint_T \kappa\, d\sigma < 0$ and so we deduce from (1) that $\alpha + \beta + \gamma < \pi$. In particular, if $\kappa = 0$ then we deduce from (1) that $\alpha + \beta + \gamma = \pi$.

Exercise 75. Show that on a simply connected surface of negative curvature two geodesics emanating from the same point will never meet.

Solution. Suppose, towards a contradiction, that a simply connected surface of negative curvature two geodesics emanating from the same point will meet. By the Gauss-Bonnet Theorem,

$$\iint_R \kappa\, d\sigma + \theta_1 + \theta_2 = 2\pi$$

where R is the simple region bounded by two acts of two consecutive intersection points, and θ_1, θ_2 are the external angles of R. Since the geodesics γ_1 and γ_2 cannot be mutually tangent, so $\gamma_i < \pi$, $i = 1, 2$. On the other hand, $\kappa < 0$, whence the contradiction (note that if $\kappa < 0$ then $\iint_R \kappa\, d\sigma < 0$).

EXERCISE 76. Let M be a surface homeomorphic to a sphere in $\mathbf{R}^3$, and let $\Gamma \subset M$ be a closed geodesic. Show that each of the two regions bounded by Γ have equal areas under the Gauss map.

SOLUTION. We may assume that the curve α is parametrized by arclength. Let $\overline{s}$ denote the arclength of the curve $n = n(s)$ on S^2. The geodesic curvature og $n(s)$ is $\overline{\kappa}_g = \left\langle \ddot{n}, n \times \dot{n} \right\rangle$ where the dots denote the differentiate with respect to $\overline{s}$. Since

$$\begin{aligned}
\dot{n} &= \frac{dn}{ds} \cdot \frac{ds}{d\overline{s}} = (-\kappa t - \tau b)\frac{ds}{d\overline{s}} \\
\ddot{n} &= (-\kappa t - \tau b)\frac{d^2 s}{d\overline{s}^2} + (-\kappa' t - \tau' b)(\frac{ds}{d\overline{s}})^2 - (\kappa^2 + \tau^2)n(\frac{ds}{d\overline{s}})^2 \\
(\frac{ds}{d\overline{s}})^2 &= \frac{1}{\kappa^2 + \tau^2},
\end{aligned}$$

so we obtain

$$\begin{aligned}
\overline{\kappa}_g &= \left\langle \ddot{n}, n \times \dot{n} \right\rangle \\
&= \frac{ds}{d\overline{s}} \left\langle \kappa t - \tau b, \ddot{n} \right\rangle \\
&= (\frac{ds}{d\overline{s}})^3(-\kappa t' + \kappa' \tau) \\
&= -\frac{\tau' \kappa - \kappa' \tau}{\kappa^2 + \tau^2} \cdot \frac{ds}{d\overline{s}} \\
&= -\frac{d}{ds} \tan^{-1}(\frac{\tau}{\kappa}) \cdot \frac{ds}{d\overline{s}}.
\end{aligned}$$

Applying the Gauss-Bonnet Theorem to one of the regions bounded by $n(I)$ and using the fact that $\kappa = 1$ to obtain

$$2\pi = \iint_R \kappa \, d\sigma + \int_{\partial R} \overline{\kappa}_g \, d\overline{s} = \text{Area}(R).$$

Since the area of S^2 is 4π, so each of the two regions bounded by Γ have equal areas under the Gauss map.

EXERCISE 77. Compute the area of the pseudo-sphere, i.e. the surface of revolution obtained by rotating a tractrix.

SOLUTION (1). The equation of the tractrix $\alpha\colon (0,\pi) \to \mathbf{R}^2$ is given by

$$\alpha(t) = (\sin t, \cos t + \log(\tan \frac{t}{2})).$$

The area of the surface of revolution around the y-axis is

$$S = \int_0^{\pi} 2\pi x \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.$$

Note that

$$\begin{aligned}
x'(t) &= \cos t \\
y'(t) &= -\sin t + \frac{\frac{1}{2}\frac{1}{\cos^2 \frac{t}{2}}}{\tan \frac{t}{2}} \\
&= -\sin t + \frac{1}{2} \cdot \frac{1}{\cos^2 \frac{t}{2}} \cdot \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \\
&= -\sin t + \frac{1}{\sin t}
\end{aligned}$$

Thus

$$\begin{aligned}
[x'(t)]^2 + [y'(t)]^2 &= \cos^2 t + \sin^2 t + \frac{1}{\sin^2 t} - 2 \\
&= \frac{1}{\sin^2 t} - 1 && \text{since } \cos^2 t + \sin^2 t = 1 \\
&= \frac{1 - \sin^2 t}{\sin^2 t} \\
&= \frac{\cos^2 t}{\sin^2 t} \\
&= \cot^2 t.
\end{aligned}$$

Hence

$$\begin{aligned}
S &= 2\pi \int_0^{\pi} \sin t \cdot |\cot t| \, dt \\
&= 2\pi \int_0^{\pi/2} \sin t \frac{\cos t}{\sin^2 t} \, dt - 2\pi \int_{\pi/2}^{\pi} \sin t \frac{\cos t}{\sin t} \, dt \\
&= 2\pi \int_0^{\pi/2} \cos t \, dt - 2\pi \int_{\pi/2}^{\pi} \cos t \, dt \\
&= 2\pi \sin t|_0^{\pi/2} - 2\pi \sin t|_{\pi/2}^{\pi} \\
&= 2\pi - 2\pi(0 - 1) \\
&= 4\pi.
\end{aligned}$$

SOLUTION (2). The parametrization equation of the surafce of revolution obtained by rotating a tractrix is

$$x(u, v) = (\sin v, \cos u, \sin v \sin u, \cos v + \log \tan \frac{t}{2}).$$

We have that

$$\begin{aligned}
x_u &= (-\sin v \sin u, \sin v \cos u, 0) \\
x_v &= (\cos v \cos u, \cos v \sin u, -\sin v + \frac{1}{\sin v}).
\end{aligned}$$

Thus

$$
\begin{aligned}
E &= \langle x_u, x_u \rangle \\
&= \sin^2 v(\sin^2 u + \cos^2 u) \\
&= \sin^2 v, && \text{since } \sin^2 v + \cos^2 v = 1,
\end{aligned}
$$

$$
F = \langle x_u, x_v \rangle = 0,
$$

$$
\begin{aligned}
G &= \langle x_v, x_v \rangle \\
&= \cos^2 v(\cos^2 u + \sin^2 u) + \sin^2 v + \frac{1}{\sin^2 v} - 2 \\
&= \cos^2 v + \sin^2 v + \frac{1}{\sin^2 v} - 2 && \text{since } \sin^2 v + \cos^2 v = 1 \\
&= \frac{1}{\sin^2 v} - 1 = \frac{\cos^2 v}{\sin^2 v}.
\end{aligned}
$$

Thus the area of the surface of revolution obtained by rotating by a tractrix is

$$
\begin{aligned}
S &= \int_{v=0}^{\pi} \int_{u=0}^{2\pi} \sqrt{EG - F^2}\, du\, dv \\
&= \int_{v=0}^{\pi} \int_{u=0}^{2\pi} |\cos v|\, du\, dv \\
&= \int_{v=0}^{\pi/2} \int_{u=0}^{2\pi} \cos v\, du\, dv - \int_{v=\pi/2}^{\pi} \int_{u=0}^{2\pi} \cos v\, du\, dv \\
&= 2\pi \sin v|_0^{\pi/2} - 2\pi \sin v|_{\pi/2}^{\pi} \\
&= 4\pi.
\end{aligned}
$$

Part 2

Manifolds

Exercise 78 (Product Manifolds). If M and N are manifolds of dimension m and n respectively, show that $M \times N$ is a manifold of dimension $m+n$, with respect to its product topology. In particular, the torus T^n is an n-dimensional manifold.

Solution. Consider two distinct points (u, v) and (r, s) in $M \times N$. Without loss of generality, we assume that $u \neq r$. Since M is Hausdorff, it follows that there exists two disjoint open subsets U_u and U_r in M containing u and r, respectively. Thus $U_u \cap U_r = \emptyset$. Let V_v and V_s be the open neighborhoods in N of v and s, respectively. The subsets $U_u \times V_v$ and $U_r \times V_s$ are disjoint open subsets in $M \times N$ containing (u, v) and (r, s), respectively. Hence, $M \times N$ is Hausdorff.

Since M and N are manifolds, so they have countable bases $\mathscr{B}_M$ and $\mathscr{B}_N$, respectively. Therefore, $M \times N$ have countable bases $\mathscr{B}_M$ and $\mathscr{B}_N = \{U \times V \mid U \in \mathscr{B}_M, V \in \mathscr{B}_N\}$.

Since M is a manifold, so for every $x \in M$, there is a neighborhood U_x which is homeomorphic to $\mathbb{R}^n$ (The dimension of M is n) under the mapping f_x.

Similarly, for every $y \in N$, there is a neighborhood V_y in N which is homeomorphic to $\mathbb{R}^m$ (the dimension of N is m) under the mapping g_y. The map $h_{x,y} \colon U_x \times V_y \to \mathbb{R}^{m+n}$ is defined by $h_{x,y}(u, v) = (f_x(u), g_y(v))$ for $(u, v) \in U_x \times V_y$. Then $U_x \times V_y$ is homeomorphic to $\mathbb{R}^{m+n}$ under the mapping $h_{x,y}$ since f_x and g_y are homeomorphisms and the dimension of the manifold $M \times N$ is $m+n$.

Exercise 79 (Group Actions). Show that if a group G acts properly discontinuosly on a manifold M, then M/G is a manifold. (*Hints:* Openness of π ensures that M/G has a countable basis. Condition (i) in the defintion of proper discontinuity ensures that π is locally one-to-one, which together with openness, yields that M/G is locally homeomorphic to $\mathbf{R}^n$. Finally, condition (ii) implies that M/G is hausdorf.)

Solution. Assume that $[p] \neq [q]$ for $[p]$ and $[q]$ in M/G. Since $[p] \neq [q]$, so $p \neq h_g(q)$ for any $g \in G$ (h_g is a homeomorphism on M). Since G acts properly discontinuous on M, so there exists open neighborhoods U and V, respectively, such that $U \cap h_g(V) = \emptyset$ for all $g \in G$ (condition (ii) in the defintion of the properly discontinuous). Therefore, $U \cap (\bigcup_{g \in G} h_g(V)) = \emptyset$. This implies that

$$\pi(U) \cap (\bigcup_{g \in G} h_g(V)) = \emptyset.$$

Since π is open and $h_g(g \in G)$ are homeomorphisms so $\pi(U)$ and $\bigcup_{g\in G} h_g(V)$ are open subsets in M/G. It is clear that $[p] \in \pi(U)$ and $[q] \in \bigcup_{g\in G} h_g(V)$. Therefore, $\pi(U)$ and $\bigcup_{g\in G} h_g(V)$ are open neighborhoods of $[p]$ and $[q]$, respectively. Thus M/G is Hausdorff.

Since M is a manifold, so M has a countable basis $\mathscr{B}_M$. We shall prove that $\pi(\mathscr{B}_M) = \{\pi(U) \mid U \in \mathscr{B}_M\}$ is a countable basis of M/G. Indeed, for any open subset V in M/G, then $\pi^{-1}(V)$ is open in M (since π is continuous). Since $\mathscr{B}_M$ is a countable basis of M, so there exists $U \in \mathscr{B}_M$ such that $U \subset \pi^{-1}(V)$. Thus $\pi(U) \subset V$. Therefore, $\pi(\mathscr{B}_M)$ is a countable basis in M/G.

For each $p \in M$, there is an open neighborhood V of p which is homeomorphic to $\mathbb{R}^n$.

$$\begin{array}{ccc} V & \xrightarrow{\pi} & M/G \\ \downarrow & \swarrow_{f} & \\ \mathbb{R}^n & & \end{array}$$

Let g be the homeomorphism from V to $\mathbb{R}^n$. Then $f\colon \pi(V) \subset M/G \to \mathbb{R}^n$ is defined by $f(\pi(v)) = g(v)$ where $v = \pi(v) \backslash \{h_g(v) \mid g \in G\backslash\{e\}\}$.

Since π is an open mapping, so $\pi(V)$ is open in M/G and it contains $[p]$. Thus $\pi(V)$ is an open neighborhood of $[p]$ in M/G. Since G acts properly discontinuous on M, so π is locally one-to-one (condition (i)). Thus f defined above is a homeomorphism from $\pi(V)$ onto $\mathbb{R}^n$. Thus M/G is a manifold.

Exercise 80. Show that $\mathbf{RP}^n$ is homeomorphic to $\mathbf{S}^n/\{\pm 1\}$, so it is a manifold.

Solution. We have

$\mathbf{RP}^n = \{(x, -x) \mid x \in \mathbf{S}^n\}$,

$\mathbf{S}^n/\{\pm 1\} = \{(f_1(x), f_{-1}(x)) \mid x \in \mathbf{S}^n, f_i(i = \pm 1)$ is the homeomorphism on $\mathbf{S}^n\}$.

The mapping $g\colon \mathbf{RP}^n \to \mathbf{S}^n/\{\pm 1\}$ is defined by

$$g((x, -x)) = (f_1(x), f_{-1}(x))$$

Since f_1 and f_{-1} are homeomorphisms on $\mathbf{S}^n$, so g is a homeomorphism from $\mathbf{S}^n$ onto $\mathbf{S}^n/\{\pm 1\}$. Since $\mathbf{RP}^n$ is a manifold and $\mathbf{RP}^n$ is homeomorphic to $\mathbf{S}^n/\{\pm 1\}$, so $\mathbf{S}^n/\{\pm 1\}$ is a manifold.

Exercise 81 (Hopf Fibration). Note that, if $\mathbf{C}$ denotes the complex plane, then $\mathbf{S}^1 = \{z \in \mathbf{C} \mid \|z\| = 1\}$. Thus, since $\|zw\| = \|z\|\,\|w\|$, $\mathbf{S}^1$ admits a natural group structure. Further, note that $\mathbf{S}^3 = \{(z_1, z_2) \mid \|z_1\|^2 + \|z_2\|^2 = 1$. Thus, for every $w \in \mathbf{S}^1$, we may define a mapping $f_w \colon \mathbf{S}_3 \to \mathbf{S}_3$ by $f_w(z_1, z_2) := (wz_1, wz_2)$. Show that this defines a group action on $\mathbf{S}_3$, and $\mathbf{S}_3/\mathbf{S}_1$ is homeomorphic to $\mathbf{S}_2$.

Solution. The unit element of the group $\mathbf{S}^1$ is 1. We have

$$\begin{aligned} f_1(z_1, z_2) &= \{1z_1, 1z_2\} \\ &= \{z_1, z_2\} \qquad\qquad \text{for } (z_1, z_2) \in \mathbf{S}^1. \end{aligned}$$

Thus f_1 is the identity function on $\mathbf{S}^1$. For $w_1, w_2 \in \mathbf{S}^1$ and $(z_1, z_2) \in \mathbf{S}^3$, we have

$$\begin{aligned} f_{w_1} \circ f_{w_2}(z_1, z_2) &= f_{w_1}(w_2 z_1, w_2 z_2) \\ &= ((w_1 w_2) z_1, (w_1 w_2) z_2) \\ &= f_{w_1 w_2}((z_1, z_2)). \end{aligned}$$

Hence $f_{w_1} \circ f_{w_2}((z_1, z_2)) = f_{w_1 w_2}((z_1, z_2))$. Therefore the group $\mathbf{S}^1$ acts on $\mathbf{S}^3$.

Consider the mapping $f \colon \mathbf{S}^3/\mathbf{S}^1 \to \mathbf{S}^2$ is given by

$$f([z_1, z_2]) = (2z_1\overline{z}_2, \|z_1\|^2 - \|z_2\|^2).$$

Assume that $f([z_1, z_2]) = f([u_1, u_2])$. Then this is equivalence to

$$\begin{cases} z_1\overline{z}_2 = u_1\overline{u}_2 & (1) \\ \|z_1\|^2 - \|z_2\|^2 = \|u_1\|^2 - \|u_2\|^2 & (2) \end{cases}$$

It follows from (1) that

$$\frac{z_1}{u_1} = \frac{\overline{u}_2}{\overline{z}_2} = v \qquad \text{for } u_1 \neq 0, z_2 \neq 0.$$

Hence

$$z_1 = vu_1, \qquad u_2 = \overline{v}z_2. \tag{3}$$

It follows from (2) that

$$\|z_1\|^2 - \|u_1\|^2 = \|z_2\|^2 - \|u_2\|^2 . \tag{4}$$

Substituting (3) into (4) gives that

$$\|v\|^2 \|u_1\|^2 - \|u_1\|^2 = \|z_2\|^2 - \|v\|^2 \|z_2\|^2 \qquad \text{since } \|v\| = \|\overline{v}\| .$$

Thus

$$\|u_1\|^2 (\|v\|^2 - 1) = \|z_2\|^2 (1 - \|v\|^2).$$

This only happens as $\|v\| = 1$ or $v = w \in \mathbf{S}^1$. Therefore, $z_1 = wu_1$, and substituting this into (2) gives

$$wu_1\overline{z}_2 = u_1\overline{u}_2.$$

This implies $z_2 = \overline{wu}_2$, so $z_2 = wu_2$.

If $u_1 = 0$, then it follows from (1) and (2) that $z_1 = 0$ and $\|z_2\| = \|u_2\|$, that is $z_2 = w'u_2$, $w' \in \mathbf{S}^1$.

If $z_2 = 0$, then it follows from (1) and (2) that $u_2 = 0$ and $\|z_1\| = \|u_1\|$, that is $z_1 = w'u_1$, $w' \in \mathbf{S}^1$.

Hence $[z_1, z_2] = [u_1, u_2]$. Thus f is an injective.

We shall prove that f is well-defined. Indeed, if $(u_1, u_2) \sim (s_1, s_2)$, then $u_1 = ws_1$ and $u_2 = ws_2$ for some $w \in \mathbf{S}^1$. We have

$$\begin{aligned} f([u_1, u_2]) &= (2u_1\overline{u}_2, \|u_1\|^2 - \|u_2\|^2) \\ &= (2ws_1\overline{ws}_2, \|w\|^2 \|s_1\|^2 - \|w\|^2 \|u_2\|^2) \\ &= (2s_1\overline{s}_2, \|s_1\|^2 - \|s_2\|^2) \qquad \text{since } w\overline{w} = \|w\|^2 = 1 \\ &= f([s_1, s_2]). \end{aligned}$$

Hence f is well-defined.

We next prove that f is surjective. Indeed, for arbitrary $(z,t) \in \mathbf{S}^2$, $(z \in \mathbf{C}, t \in \mathbf{R})$, we have $z = re^{iw_0}$ for $w_0 = \mathrm{Arg}(z)$, where $r = \sqrt{1-t^2} = \|z\|$ (since $\|z\|^2 + t^2 = 1$). Consider

$$\begin{aligned} z_1 &= \sqrt{\frac{1+t}{2}}w, \\ z_2 &= \sqrt{\frac{1-t}{2}}we^{iw_0}, \end{aligned}$$

where $w \in \mathbf{S}^1$. Then $(z_1, z_2) \in \mathbf{S}^3/\mathbf{S}^1$ and

$$\begin{aligned} f([z_1, z_2]) &= (2z_1\overline{z}_2, \|z_1\|^2 - \|z_2\|^2) \\ &= (\sqrt{1-t^2}e^{iw_0}, \frac{1+t}{2} - \frac{1-t}{2}) \quad \text{since } \|w\| = 1, \left\|e^{iw}\right\| = 1. \\ &= (z,t). \end{aligned}$$

Thus f is surjective. So f is bijective.

It is clear that f is continuous and $\mathbf{S}^3/\mathbf{S}^1$ is compact and f is a bijective from a compact set onto a Hausdorff space $\mathbf{S}^2$, therefore f is a homeomorphism. That is to say $\mathbf{S}^3/\mathbf{S}^1$ is homeomorphic to $\mathbf{S}^2$.

Exercise 82. Show that $\mathbf{S}^n$ is a smooth (C^∞) manifold.

Solution. Let

$$U_i = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_i = 0, x_1^2 + x_2^2 + \cdots + x_{i-1}^2 + x_{i+1}^2 + \cdots x_{n+1}^2 < 1\},$$

$$\varphi_i((x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})) = (x_1, x_2, \ldots, x_{i-1}, D_i, x_{i+1}, \ldots, x_{n+1}) \in \mathbf{R}^{n+1},$$

where $D_i = \sqrt{1 - (x_1^2 + \cdots x_{i-1}^2 + x_{i+1}^2 + \cdots x_{n+1}^2)}$, and

$$\psi_i((x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_{n+1})) = (x_1, \ldots, x_{i-1}, -D_i, x_{i+1}, \ldots x_{n+1}).$$

It is clear that $\varphi_i \circ \varphi_j^{-1}$, $\psi_i \circ \psi_j^{-1}$, $\varphi_i \circ \psi_j^{-1}$, $\psi_i \circ \varphi_j^{-1}$ are the smooth (C^∞) functions.

We also have

$$\bigcup_{i=1}^{n+1} U_i = \mathbf{S}^n.$$

The family $\{(U_i, \varphi_i), (U_i, \psi_i), i = \overline{1, n+1}\}$ is an atlas of the manifold $\mathbf{S}^n$. The functions φ_i and ψ_i are homeomorphisms. Therefore $\mathbf{S}^n$ is a smooth (C^∞) manifold.

EXERCISE 83. Show that the notion of smoothness of a function $f\colon M \to N$ is well-defined (i.e., it is independent of the choice of local charts).

SOLUTION. Assume that for evry $p \in M$ there exists other local charts (U', φ') of M and (V', ψ') of N, centered at p and $f(p)$ respectively. Since M has a differential structure, so let $U_1 = U \cap U'$, $V_1 = V \cap V'$, then $(U_1, \varphi'|_{U_1})$ and $(V_1, \psi'|_{V_1})$ are local charts, centered at p and $f(p)$, respectively.

We have $\psi'|_{V_1} \circ f \circ \varphi'|_{U_1} = \psi \circ f \circ \varphi^{-1}$ is smooth (since $(U_1, \varphi'|_{U_1}) \subset (U, \varphi)$ and $(V_1, \psi'|_{V_1}) \subset (V, \psi)$). Therefore the notion of smoothness of a function $f\colon M \to N$ is well-defined.

EXERCISE 84 (Exercise 0.2 from [Car92].). Prove that the tangent bundle of a differential manifold M is orientable (even though M may not be).

SOLUTION. Let $\{(U_\alpha, x_\alpha)\}$ be any atlas of M, and $\hat{A}$ be the atlas of TM given by $\{(V_\alpha = \pi^{-1}(U_\alpha), \hat{x}_\alpha = (x_\alpha \times id) \circ \varphi_\alpha)\}$, where $\varphi_\alpha\colon V_\alpha \to U_\alpha \times \mathbf{R}^n$ are the trivializing maps. We shall prove that this is an oriented atlas.

Define $\hat{\phi}_\alpha(u, v) = \hat{x}_\beta \circ \hat{x}_\alpha^{-1}$.

For a vector $(u, v) \in \mathbf{R}^n \times \mathbf{R}^n$, then $\hat{\phi}(u, v) = (\phi(u), A(u) \cdot v)$, where $\phi = x_\beta \circ x_\alpha^{-1}$, and $A_{ij}(u) = \frac{\partial \phi^i}{\partial u^j}|_u$.

Let $(u^1, \dots, u^n, v^1, \dots, v^n)$ be coordinates around $(u, v) \in \mathbf{R}^n \times \mathbf{R}^n$. Given $i, j = 1, 2, \dots, n$, one has

$$\begin{aligned}
\frac{\partial \phi^i}{\partial u^j}|_{(u,v)} &= \frac{\partial \phi^i}{\partial x^j} = A_{ij}(u), \\
\frac{\partial \phi^i}{\partial v^j}|_{(u,v)} &= 0, \\
\frac{\partial \phi^{n+i}}{\partial v^j}|_{(u,v)} &= \frac{\partial}{\partial v^j}|_{(u,v)}(A(u) \cdot v) = A_{ij}(u).
\end{aligned}$$

Therefore, the Jacobian matrix of $\hat{\phi}$ has the block form

$$\begin{pmatrix} A & * \\ 0 & A \end{pmatrix}$$

and its determinant equals $\det(A)^2 > 0$. Thus TM is orientable.

EXERCISE 85 (Exercise 0.5 (Embedding of $P^2(\mathbf{R})$ in $\mathbf{R}^4$) from [Car92].). Let $F\colon \mathbf{R}^3 \to \mathbf{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz), \qquad (x, y, z) = p \in \mathbf{R}^3.$$

Let $S^2 \subset \mathbf{R}^3$ be the unit sphere with the origin $0 \in \mathbf{R}^3$. Observe that the restriction $\varphi = F|S^2$ í such that $\varphi(p) = \varphi(-p)$, and consider the mapping $\tilde{\varphi}\colon P^2(\mathbf{R}) \to \mathbf{R}^4$ given by

$$\tilde{\varphi}([p]) = \varphi(p), \qquad [p] = \text{equiv. class of } p = \{p, -p\}.$$

Prove that

(a) $\widetilde{\varphi}$ is an immersion.

(b) $\widetilde{\varphi}$ is injective; together with (a) and the compactness of of $P^2(\mathbf{R})$, this implies that $\widetilde{\varphi}$ is an embedding.

SOLUTION. (b) We shall prove that $\widetilde{\varphi}$ is an embedding of $P^2(\mathbf{R})$ into $\mathbf{R}^4$. We first prove $\widetilde{\varphi}$ is injective. Indeed, assume that $\widetilde{\varphi}([p]) = \widetilde{\varphi}([q])$ for $p = (x, y, z)$ and $q = (u, v, w)$ in $\mathbf{R}^3$. This is equivelent to

$$\begin{cases} x^2 - y^2 & = u^2 - v^2, \qquad (1) \\ xy & = uv, \qquad (2) \\ xz & = uw, \qquad (3) \\ yz & = vw. \end{cases}$$

Since $P^2(\mathbf{R})$ is the quotient of $\mathbf{S}^2$ by the equivalent relation that identifies $p \in \mathbf{S}^2$ with its anipodal point, so $\|p\| = \|q\| = 1$, that is $x^2 + y^2 + z^2 = u^2 + v^2 + w^2 = 1$. Combine this together with (1) and (3) to obtain

$$\begin{cases} x^2 + x^2 & = u^2 + w^2 \\ xz & = uw \end{cases}$$

This system has solutions

$$\begin{cases} x & = u \\ z & = w \end{cases}$$

or

$$\begin{cases} x &= -u \\ z &= -w \end{cases}$$

Combine these with (2) to obtain

$$\begin{cases} x &= u \\ y &= v \\ z &= w \end{cases}$$

or

$$\begin{cases} x &= -u \\ y &= -v \\ z &= -w \end{cases}$$

That is $p \sim q$. Then $[p] = [q]$, and hence $\widetilde{\varphi}$ is injective.

Since $P^2(\mathbf{R})$ is a compact set and $\widetilde{\varphi}$ is injective from $P^2(\mathbf{R})$ into the Hausdorff space $\mathbf{R}^4$, so $\widetilde{\varphi}$ is an embedding of $P^2(\mathbf{R})$ in $\mathbf{R}^4$.

(a) But an embedding is also an emersion, hence $\widetilde{\varphi}$ is an immersion.

Exercise 86 (Exercise 0.8 from [Car92].). Let M_1 and M_2 ve differential manifolds. Let $\varphi\colon M_1 \to M_2$ be a local diffeomorphism. Prove that if M_2 is orientable, then M_1 is orientable.

Solution. Assume that M_2 is orientable and it admits a differentiable structure $\{(U_\alpha, x_\alpha)\}$. At each point $p \in M_2$, there exists a neighborhood V of p and W of $\varphi(p)$ such that V and W are diffeomeomorphic under the mapping φ.

Let $y_\alpha\colon \mathbf{R}^n \to M_1$ be given by $y_\alpha = \varphi^{-1} \circ x_\alpha$.

The differential structure $\{(V_\alpha, y_\alpha)\}$ of M_1 satisfies

$$\begin{aligned} y_\alpha^{-1} \circ y_\beta &= y_\alpha^{-1} \circ \varphi^{-1} \circ x_p \\ &= x_\alpha^{-1} \circ \varphi \circ \varphi^{-1} \circ \beta \\ &= x_\alpha^{-1} \circ x_\beta && \text{since } \varphi \circ \varphi^{-1} = id. \end{aligned}$$

But M_2 is orientable, so the differential of $x_\alpha^{-1} \circ x_\beta$ has positive determinant. This implies that the differential of $y_\alpha^{-1} \circ y_\beta$ has positive determinant. Thus M_1 is orientable.

Exercise 87 (Exercise 0.9 from [Car92].). Let $G \times M \to M$ be a properly discontinuous action of a group G on a differentiable manifold M.

(a) Prove that the manifold M/G is orientable if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.

(b) Use (a) to show that the projective plane $P^2(\mathbf{R})$, the Klein bottle and the Mobius band are non-orientable.

(c) Prove that $P^n(\mathbf{R})$ is orientable if and only if n is odd.

Solution. (a) Let $\pi\colon M \to M/G$ ve given by

$$\pi(p) = [p] \qquad\qquad \text{for } p \in M.$$

M is orientable if and only if there exist a chart $\{(U_\alpha, x_\alpha)\}$ such that for every pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$, the differential of $x_\alpha^{-1} \circ x_\beta$ has positive determinant.

Note that the family $\{(\pi(U_\alpha), \pi \circ x_\alpha)\}$ is a differntial structure of M/G. (Note that $x_\alpha\colon U_\alpha \subset \mathbf{R}^n \to M$ and π is an open mapping).

We have that

$$\begin{aligned}(\pi \circ x_\alpha)^{-1} \circ (\pi \circ x_\beta) &= x_\alpha^{-1} \circ \pi^{-1} \circ \pi \circ x_\beta \\ &= x_\alpha^{-1} \circ x_\beta.\end{aligned}$$

The differential of $(\pi \circ x_\alpha)^{-1} \circ (\pi \circ x_\beta)$ has positive determinant if and only if the differential of $x_\alpha^{-1} \circ x_\beta$ has positive determinant. Thus M/G is orientable if and only if M is orientable.

(b) We first show that Mobius strip is not orientable. Consider M is the right circular cylinder given by $C = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 = 1, -1 < z < 1\}$. The diffeomorphisms of M are formed by $\{A, id\}$, where $A(p) = -p$ for $p \in M$. Then M/G is a Mobius strip. Since M is non-orientable, so by applying (a) we obtain that the Mobius strip is non-orientable.

Since $P^2(\mathbf{R})$ contains a Mobius strip and Mobius strip is non-orientable, so $P^2(\mathbf{R})$ is non-orientable since it admits an atlas such that there exists a pair of charts (U_α, x_α), (U_β, x_β) of M, $x_\alpha^{-1} \circ x_\beta$ is not orientation preserving.

When M is the torus of revolution T^2, then M/G is is a Klein bottle and we know that M is non-orientable, so M/G is non-orientable by (a).

(c) Recall that we have a natural mapping $\pi\colon \mathbf{S}^n \to \mathbf{RP}^n$ which is a local diffeomorphism, and is given simply by $\pi(p) = \{p, -p\}$. Now let $r\colon S^n \to \mathbf{S}^n$ be the reflection through the origin. Then

$$\pi \circ r = \pi.$$

If $\mathbf{RP}^n$ is orientable then we may assume that π preserves orientation. Then the above inequality implies that $\pi \circ r$ preserves orientation as well. This is not possible only if r preserves orientation which is the case only when n is odd. Thus $\mathbf{RP}^n$ is not orientable when n is even.

It remains to show that $\mathbf{RP}^n$ is orientable when n is odd. In this we may orient each tangent space $T_{[p]}\mathbf{RP}^n$ is as follows: take a representative from $q \in [p] = \{p, -p\}$. Choose a basis of $T_q\mathbf{S}^n$ which is in its orientation class, and let the image of this basis under $d\pi$ determine the orientation class of $T_{[p]}\mathbf{RP}^n$. This orientation is well defined because it not effected by whether $q = p$ or $q = -p$. Indeed, in $(b_1, \ldots, b_n)$ is a basis in the orientation class of $T_p\mathbf{S}^n$ and $(b'_1, \ldots, b'_n)$ is a basis in the orientation class of $T_{-p}\mathbf{S}^n$ then

$$(d\pi_p(b_1), \ldots, d\pi(b_n)) \qquad \text{and} \qquad (d\pi_{-p}(b'_1), \ldots, d\pi(b'_n))$$

belong in the same orientation class of $T_{[p]}\mathbf{RP}^n$; because

$$d\pi_p(b_i) = d(\pi \circ r)_p(b_i) = d\pi_{r(p)} \circ dr_p(b_i) = d\pi_{-p} \circ dr_p(b_i)$$

and r preserves orientation, i.e., $(dr_p(b_1), \ldots, dr_p(b_n))$ belongs in the same orientation class as $(b'_1, \ldots, b'_n)$.

Exercise 88. Show that O_n is a smooth $n(n-1)/2$-dimensional submanifold of GL_n. (*Hint:* Define $f\colon GL_n \to GL_n$ by $f(A) := A^TA$. Then show that T_AGL_n is given by the equivalence class of curves of the form $A + tB$ where B is any $n \times n$ matrix. Finally, show that $df_A(T_AGL_n)$ is isomorphic to the space of symmetric $n \times n$ matrices).

Solution. Consider the mapping $f\colon GL_n \to GL_n$ given by

$$f(A) = A^TA$$

for any $A \in GL_n$. We first note that $A \in O_n$ is equivalence to $A^TA = I$ where I is the identity matrix. But $\det(A^TA) = A^2$ and $\det I = 1$, so $\det A = \pm 1$. This implies that $O_n = f^{-1}(I)$. f is the composition of C^∞-maps, so f is C^∞.

Since $[f(A)]^T = (A^TA)^T = A^TA = f(A)$, so $f(A)$ is a symmetrix matrix.

We have

$$\begin{aligned} df_A(B) &= \lim_{t\to 0} \frac{(A+tB)^T(A+tB) - A^T A}{t} \\ &= \lim_{t\to 0} \frac{(A^T+tB^T)(A+tB) - A^T A}{t} \\ &= A^T B + B^T A \\ &= A^T B + (A^T B)^T. \end{aligned}$$

df_A is a composition of two maps

$$\begin{aligned} \varphi\colon GL_n &\to GL_n \\ B &\mapsto A^T B, \\ \psi\colon GL_n &\to Sym(n) \\ B &\mapsto B + B^T, \end{aligned}$$

where $Sym(n)$ is the set of invertible symmetric matrices. φ is onto since A^T is invertible and ψ is onto since for any $C \in Sym(n)$, then $\psi(\frac{1}{2}C) = C$. Thus $df_A = \psi \circ \varphi$ is surjective and has rank $\frac{n(n+1)}{2}$.

Applying Theorem 2 of Lecture Note 7 to obtain that $f^{-1}(I) = O_n$ is a smooth submanifold of GL_n and it has dimensional $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. Thus O_n is a smooth $\frac{n(n-1)}{2}$ dimensional submanifold of GL_n.

Exercise 89 (Exercise 1.1 from [Car92].). Prove that the antipodal mapping $A : S^n \to S^n$ given by $A(p) = -p$ í an ísometry of S^n. Use this fact to introduce a Riemannian metric on the real projective space $P^n(R)$ such that the natural projection $\pi : S^n \to P^n(\mathbf{R})$ is a local isometry

Solution. Since $A(p) = -p$, so

$$\begin{aligned} dA_p(q) &= -\lim_{t\to 0} \frac{A(p+tq) - A(p)}{t} \\ &= \lim_{t\to 0} \frac{-p - dq - (-p)}{t} \\ &= -q. \end{aligned}$$

Thus $aA_p(q) = -q$. Thereofore

$$\begin{aligned}\langle u, v\rangle_p &= \langle -u, -v\rangle_{-p} && \text{since the inner product is bilinear}\\ &= \langle dA_p(u), dA_p(v)\rangle_{A(v)}.\end{aligned}$$

Hence A is an isometry of $\mathbf{S}^n$. Define an inner product on $P^n(\mathbf{R})$ by

$$\langle [p], [q]\rangle = |\langle p.q\rangle|$$

for $p.q \in \mathbf{S}^n$. This inner product is well-defined because it is not depend on the representation of the class.

At each point $p \in \mathbf{S}^n$, there is a neighborhood U such that for any $u, v \in U$, then $\langle u, v\rangle \geq 0$ since $\langle p, p\rangle \geq 0$ and the inner product on $\mathbf{S}^n$ is continuous. Thus for $u, v \in U$, we have

$$\begin{aligned}\langle u, v\rangle_{[p]} &= \left|\langle -u, -v\rangle_p\right|\\ &= \left|\langle dA_p(u), dA_p(v)\rangle_{A(p)}\right|\\ &= \left|\langle d\pi_p(u), d\pi_p(v)\rangle_{A(p)}\right|\\ &= \langle d\pi_p(u), d\pi_p(v)\rangle_{\pi(p)} && \text{since } d\pi_p(u) = \pi(u) \text{ and } A_p(u) \in [u]\end{aligned}$$

Thus this inner product is a Riemann metric on $P^n(\mathbf{R})$ and π is a local isometry.

Exercise 90 (Exercise 1.3 from [Car92].). Obtain an isometric immersion of the flat torus T^n into $\mathbf{R}^{2n}$

Solution. We have $T^n = S^1 \times S^1 \times \cdots \times S^1$ (The Cartesian product of n coppies of S^1).

Consider the mapping

$$\psi \colon T^n \to \mathbf{R}^{2n}$$

is defined by

$$\psi(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_n}) = (\cos x_1, \sin x_1, \cos x_2, \sin x_2, \ldots, \cos x_n, \sin x_n)$$

where $x_i \in \mathbf{R}$, $\forall i \in \{1, 2, \ldots, n\}$.

For $p = (e^{ix_1}, \ldots, e^{ix_n}), q = (e^{iu_1}, \ldots, e^{iu_n}) \in T^n$, then

$$p + tq = (e^{ix_1} + te^{iu_1}, \ldots, e^{ix_n} + te^{iu_n}).$$

So

$$d\psi_p(q) = \lim_{t\to 0} \frac{\psi(p+tq) - \psi(p)}{t} = q.$$

Thus $d\psi_p$ is injective, and so ψ is an immersion.

We have

$$\langle u, v\rangle_p = \langle d\psi_p(u), d\psi_p(v)\rangle_{\psi(p)}$$

since $d\psi_p(q) = q$. Hence ψ is an isometric immersion.

Exercise 91. Show that the the Poincare half-plane and the half-disk are isometric (Hint: identify the Poincare half-plane with the region $y > 1$ in $\mathbf{R}^2$ and do an inversion).

Solution. Let $\|\cdot\|_p$ be the norm generated by the inner product $g_p(X,Y) = \frac{\langle X,Y\rangle}{(1-\|p\|^2)^2}$. That is $\|X\|_p = \sqrt{g_p(X,X)}$, $X \in B^n = \{X \in \mathbf{R}^n \mid \|X\|_p < 1\}$. Let $H^n = \{(x_1,\dots,x_n) \in \mathbf{R}^n \mid x_n > 0\}$. Define the mapping $f\colon B^n \to H^n$ by

$$f(p) = 2\frac{p - p_0}{\|p - p_0\|^2} - (0,0,\dots,0,1)$$

where $p_0 = (0,0,\dots,0,-1) \in \mathbf{R}^n$, $p \in B^n$. For v is a vector at p and $\langle\cdot,\cdot\rangle$ denotes the inner product in the Euclidean metric,

$$\langle df_p(v), df_p(v)\rangle = \frac{\langle v, v\rangle}{\|p - p_0\|^4}.$$

Let $f(p) = (f_1(p), f_2(p), \dots, f_n(p))$. Then

$$\begin{aligned}
f_n(p) &= 2\frac{p_0 + 1}{\|p - p_0\|^2} - 1\\
&= \frac{2p_n + 2 - \sum_{i=1}^n p_i^2 - 2p_n - 1}{\|p - p_0\|^2}\\
&= \frac{1 - \|p\|^2}{\|p - p_0\|^2} > 0
\end{aligned}$$

for $p = (p_1, p_2, \dots, p_n) \in B^n$.

Note that $X \in B^n$. Then $\|X\|_p^2 < 1$. so $\frac{\langle X,X\rangle}{(1-\|p\|^2)^2} < 1$. Thus $\|X\| < 1 - \|p\|^2 < 1$. We have

$$\begin{aligned} h_p(v,v) &= \frac{\langle df_p(v), df_p(v)\rangle}{[f_n(p)]^2} \\ &= \frac{\|p-p_0\|^4 \langle v,v\rangle}{(1-\|p\|^2)^2 \|p-p_0\|^4} \\ &= g_p(v,v) \end{aligned}$$

where $h_p(X,Y) = \frac{\langle X,Y\rangle}{(p_n)^2}$ is the inner product in H^n. Since f is injective, so we conclude that f is an isometry of B^n onto H^n.

EXERCISE 92. Compute the metric of the surface given by the graph of a function $f\colon \Omega \subset \mathbf{R}^2 \to \mathbf{R}$.

SOLUTION. Let $z = f(x,y)$ be the function representing the surface in $\mathbf{R}^3$. Define the function $g\colon \mathbf{R}^2 \to \mathbf{R}^3$ by

$$g(x,y) = (x,y,f(x,y)).$$

We have

$$\begin{aligned} D_1 g(x,y) &= (1,0,\frac{\partial f}{\partial x}), \\ D_2 g(x,y) &= (0,1,\frac{\partial f}{\partial y}). \end{aligned}$$

Therefore, $g_{ij}(f(x,y))$ is given by

$$\begin{pmatrix} 1 + (\frac{\partial f}{\partial x})^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + (\frac{\partial f}{\partial y})^2 \end{pmatrix}.$$

EXERCISE 93. Compute the length of the radius of the Poincare-disk (with respect to the Poincare metric).

SOLUTION. Let $B^n = \{p \in \mathbf{R}^n \mid \|p\|_B < 1\}$ where $\|p\|_B = \sqrt{g_p(p,p)}$ is the norm generated by the inner product g_p. For any $X \in B^n$, then $\|X\|_B^2 < 1$. This is equivalnce to

$$\begin{aligned} & g_p(X,X) < 1 \\ \iff \quad & \frac{\langle X, X\rangle}{(1-\|p\|^2)^2} < 1 \\ \iff \quad & \|X\| < 1 - \|p\|^2 . \end{aligned}$$

Therefore the radius of B^n is $r = 1 - \|p\|^2$.

EXERCISE 94 (Exercise 2.1 from [Car92].). Let M be a Riemannian manifold. Consider mapping
$P = P_{c,t_o,t} : T_{c(t_o)}M \to T_{c(t)}M$
defined by: $P_{c,t_o t}(v)$, $v \in T_{c(t_o)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is orientel, P preserves the orientation.

SOLUTION. The vector field V in the parallel transporting along the curve c. Since $\frac{DV}{dt} = 0$, so

$$\begin{aligned} \frac{d}{dt}\langle V(t_1), V(t_2)\rangle &= \left\langle \frac{DV(t_1)}{dt}, V(t_2)\right\rangle + \left\langle V(t_1), \frac{DV(t_2)}{dt}\right\rangle \\ &= 0 \end{aligned}$$

for all $t_1, t_2 \in I$ and $c\colon I \to M$. Therofore $\langle V(t_1), V(t_2)\rangle = \text{const}$. In particular, $\langle V(t), V(t)\rangle = \text{const}$, for $t \in I$. Thus

$$\begin{aligned} \langle P_{c,t_0,t}(V(t_0)), P_{c,t_0,t}(V(t_0))\rangle &= \langle V(t), V(t)\rangle \\ &= \langle V(t_0), V(t_0)\rangle \end{aligned}$$

or

$$\|P_{c,t_0,t}(u)\| = \|u\| ,$$

for $u \in T_{c(t_0)}M$.

Since $P_{c,t_0,t}(v)$, $v \in T_{c(t_0)}M$ is linear, so for $u, v \in T_{c(t_0)}M$,

$$\|P_{c,t_0,t}(u - v)\| = \|u - v\| ,$$

which is equivalent to

$$\|P_{c,t_0,t}(u) - P_{c,t_0,t}(v)\| = \|u - v\|\,.$$

Thus $P_{c,t_0,t}$ is a linear isometry.

Let $\{E_1, E_2, \ldots, E_n\}$ be an oriented basis at T_pM where $p = c(t_0)$. Denote $P_t = P_{c,t_0,t}$ be the parallel transporting. Since the parallel transporting preserves angles and lengths, so $\{P_tE_1, P_tE_2, \ldots, P_tE_n\}$ is an orthogonal frame along c. We have

$$d_pP_{c,t_0,t} = (P_tE_1, P_tE_2, \ldots, P_tE_n).$$

Let $f(t) = \det(P_tE_1, \ldots, P_tE_n)$. If $f(b) < 0$, where $b \in [0,1]$ (and $c\colon [0,1] \to M$ is a smooth curve), then by Picard-Lindeloft, $f(t)$ is snooth in t, and so by the intermediate value theorem, there is a $\bar{t} \in [0,a]$ such that $f(\bar{t}) = 0$. This contradicts to the fact that $\{P_tE_1, \ldots, P_tE_n\}$ is an orthogonal basis. Thus, the parallel transporting preserves orientation.

Exercise 95 (Exercise 2.2 from [Car92].). Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in$ Mand let c: $I \to M$ be an integral curve of Xthrough p, i.e. $c(t_o) = p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M is

$$(\nabla_X Y)(p) = \frac{d}{dt}(P^{-1}_{c,t_0,t}(Y(c(t))))_{t=t_o}$$

where $P_{c,t_o,t}\colon T_{c(t_o)}M \to T_{c(t)}M$ is the parallel transport along c, from $t_o$to t (this shows how the connection can be reobtained from the concept of parallelism).

Solution. By Proposition 2.2 from [Car92],

$$\nabla_{\frac{dc}{dt}} Y(p) = \nabla_{X(c(t))} Y(p) = \nabla_X Y(p) = \frac{DV}{dt}|_{t_0}(t) \tag{1}$$

for $V(t) = Y(c(t))$. Moreover, since

$$P_{c,t_0,t}(V(t_0)) = V(t),$$

so

$$\begin{aligned} V(t_0) &= P^{-1}_{c,t_0,t}(V(t)) \\ &= P^{-1}_{c,t_0,t}(Y(c(t))). \end{aligned}$$

But

$$\frac{d}{dt}P^{-1}_{c,t_0,t}(Y(c(t)))|_{t_0} = \frac{dV(t_0)}{dt}|_{t_0} = 0 \qquad (2)$$

It follows from (1) and (2) that

$$\nabla_{\frac{dc}{dt}}Y(p) = \frac{d}{dt}P^{-1}_{c,t_0,t}(Y(c(t)))|_{t_0}$$

EXERCISE 96 (Exercise 2.3 from [Car92].). Let $f\colon M^n \to \overline{M}^{n+k}$ be an immersion of differentiable mani-fold Minto a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric inducel by f (cf. Example 2.5 of Chap. 1). Let $p \in M$ and let $U \subset M$ be a neighborhood of p such that $f(U) \subset \overline{M}$ is a submanifold of $\overline{M}$. Futher, suppose that X, Y are differentiable vector fields on $f(U)$ which extend to differentiable vector fields on $f(U)$ which extend to differentiable vector fields $\overline{X}, \overline{Y}$on an open set of $\overline{M}$. Define $(\nabla_X Y)(p)$ =tangential component of $\overline{\nabla}_{\overline{X}}\overline{Y}(p)$, where $\overline{\nabla}$ is the Riemannian connection of M

SOLUTION. Denote

$$\overline{\nabla}_{\overline{X}}\overline{Y}(p) = (\nabla_X Y)(p) + \overline{\nabla}^{\perp}_{\overline{X}}\overline{Y}(p)$$

where $(\nabla_X Y)(p)$, $(\overline{\nabla}^{\perp}_{\overline{X}}\overline{Y})(p)$ are the tangential component and the normal component. respectively. We have

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= \overline{\nabla}_{\overline{X}}\overline{Y}(p) - \overline{\nabla}^{\perp}_{\overline{X}}\overline{Y}(p) - \overline{\nabla}_{\overline{Y}}\overline{X}(p) + \overline{\nabla}^{\perp}_{\overline{Y}}\overline{X}(p) \\
&= [\overline{\nabla}_{\overline{X}}\overline{Y}(p) - \overline{\nabla}_{\overline{Y}}\overline{X}(p)] + [\overline{\nabla}^{\perp}_{\overline{Y}}\overline{X}(p) - \overline{\nabla}^{\perp}_{\overline{X}}\overline{Y}(p)] \\
&= [\overline{X}, \overline{Y}] - pr_1[\overline{\nabla}_{\overline{X}}\overline{Y}(p) - \overline{\nabla}_{\overline{Y}}\overline{X}(p)] \qquad \text{where } pr_1 \text{ is the projection on the normal at } p \\
&= [\overline{X}, \overline{Y}] - pr_1[[\overline{X}, \overline{Y}]] \\
&= [X, Y].
\end{aligned}$$

Thus $\nabla_X Y$ is symmetric.

For any pair of parallel vector fields P and P' along C, we have

$$\begin{aligned}
\frac{d}{dt}\langle P, P'\rangle &= \left\langle \frac{DP}{dt}, P'\right\rangle + \left\langle P, \frac{DP'}{dt}\right\rangle \\
&= 0
\end{aligned}$$

since $\frac{DP}{dt} = \frac{DP'}{dt} = 0$. Thus $\langle P, P' \rangle = \text{const}$. Therefore ∇ is compatible with the Riemann metric and so ∇ is the Riemann connection of M.

EXERCISE 97 (Exercise 2.4 from [Car92].). Let $M^2 \subset \mathbf{R}^3$ be a surface in $\mathbf{R}^3$ with the induced Riemannian metric. Let $c\colon I \to M$ be a differentiable curve on M and let V be vector field tangent to Malong c; V can be thoughr of as a smooth function $V\colon I \to \mathbf{R}^3$, with $V(t) \in T_{c(t)}M$.

(a) Show that $V$is parallel if and only if $\frac{dV}{dt}$is perpendicular to $T_{c(t)}M \subset \mathbf{R}^3$where $\frac{dV}{dt}$is the usual derivative of $V\colon I \to \mathbf{R}^3$.

(b) If $S^2 \subset \mathbf{R}^3$is the unit sphere of $\mathbf{R}^3$, show that the velocity field along great circles, parametrized by arc length, is a parallel field. A similar argument holds for $S^n \subset \mathbf{R}^{\mathrm{n}+1}$.

SOLUTION. (a) Note that on the Euclidean space $\mathbf{R}^n$, $\Gamma^k_{ij} = 0$, so $\frac{DV}{dt} = \frac{dV}{dt}$ where $V\colon I \to T_pM$. That is, the covariant derivative concides with the usual derivative of $V\colon I \to T_pM$.

We first suppose that V is parallel. We have $\frac{DV}{dt} = 0$, and so $\frac{dV}{dt} = 0$. Thus $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M$.

Conversely, suppose that $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M$. Then $\left\langle \frac{dV}{dt}, P \right\rangle = 0$ for any $P \in T_{c(t)}M$. This implies that $\left\langle \frac{DV}{dt}, P \right\rangle = 0$ for any $P \in T_{c(t)}M$ (since $\frac{DV}{dt} = \frac{dV}{dt}$). In particular, if $P = \frac{DV}{dt}$, then we obtain

$$\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle = 0.$$

Thus $\frac{DV}{dt} = 0$. Therefore, V is parallel.

(b) WLOG, we consider the great circle

$$c(\theta, 0) = (\sin\theta, 0, \cos\theta).$$

The velocity vector at $p = (\sin\theta, 0, \cos\theta)$ is

$$\dot{c}(\theta, 0) = (\cos\theta, 0, -\sin\theta).$$

We have $\Big\langle c(\theta, 0), \dot{c}(\theta, 0) \Big\rangle = \sin\theta\cos\theta - \cos\theta\sin\theta = 0$. Thus $\frac{dc}{d\theta}$ is perpendicular to T_pS^2, so the velocity field is a parallel field by (a).

For $S^n = \{(x_1, \ldots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$, then as $x_{n+1} = 0$ we obtain the great circle $F^n = \{(x_1, \ldots, x_n, 0) \mid \sum_{i=1}^{n} x_i^2 = 1\}$. Consider $V = (x_1, \ldots, x_n, 0) \in F^n$. Then $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Thus

$$x_1\dot{x_1} + \cdots + x_n\dot{x_n} = 0. \qquad (1)$$

The tangent plane surface at $p = (\xi_1, \ldots, \xi_n)$ is

$$\sum_{i=1}^{n}(x_i - \xi_i)\xi_i = 0.$$

For any $(x_1, \ldots, x_n) \in T_pS^n$, it follows from (1) that

$$\left\langle (x_1, \ldots, x_n), \frac{dV}{dt} \right\rangle = 0.$$

Thus $\frac{dV}{dt}$ is perpendicular to T_pS^n. Therefore the velocity field is parallel.

EXERCISE 98 (Exercise 2.5 from [Car92].). In Euclidean space, the parallel transport of a vector between two points does not depend on the curve joining the two points.Show, by example, that this fact may not be true on an arbitrary Riemannian manifold

SOLUTION. Note that in the Euclidean space, $\frac{DV}{dt} = \frac{dV}{dt}$, where $V\colon I \to T_pM$. If V is a parallel transport, then $\frac{DV}{dt} = \frac{dV}{dt} = 0$. Thus it is not depend on the curve joining the two points.

Consider the upper half plane

$$\mathbf{R}^2_+ = \{(x, y) \mid y > 0\}$$

with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$. Hence we obtain $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

Since $\Gamma^2_{12} \neq 0$ and $\Gamma^2_{22} = \Gamma^1_{12} \neq 0$, so the fact

$$\frac{DV}{dt} = \sum_k \{\frac{dv^k}{dt} + \sum_{i,j} \Gamma^k_{ij} v^j \frac{dx_i}{dt}\} X_k$$

implies $\frac{DV}{dt} \neq \frac{dV}{dt}$.

Further, consider $v_0 = (0, 1)$ be a tangent vector at point $(0, 1)$ of $\mathbf{R}^2_+$. Let $v(t)$ be the parallel tranport of v_0 along the curve $x = t$, $y = 1$. The field $v(t) = (a(t), b(t))$ satisfies

$$\begin{cases} \frac{da}{dt} + \Gamma^1_{12} b &= 0, \\ \frac{db}{dt} + \Gamma^2_{11} a &= 0. \end{cases}$$

Taking $a = \cos\theta(t)$, $b = \sin\theta(t)$ and along the given curve $y = 1$, we obtain from the equation above that $\frac{d\theta}{dt} = -1$. Since $v(0) = v_0$, so $\theta(t) = \frac{\pi}{2} - t$. This shows that the parallel transport depends on the curve joining the two points.

EXERCISE 99. Show that the bracket satisfies the following properti

$$[X, Y] = -[Y, X] \quad \text{and} \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

SOLUTION. We first prove that

$$[X, Y] = -[Y, X].$$

We have that

$$\begin{aligned} [X, Y]_p f &= X_p(Yf) - Y_p(Xf) \\ &= -[Y_p(Xf) - X_p(Yf)] \\ &= -[Y, X]_p f \end{aligned}$$

for any function f on M that is differentiable at p. Thus

$$[X, Y] = -[Y, X],$$

as required.

We now prove that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We have that

$$\begin{aligned} [X, [Y, Z]]_p f &= X_p([Y, Z]_p f) - [Y, Z]_p(Xf) \\ &= X_p(Y_p(Zf)) - X_p(Z_p(Yf)) - Y_p(Z_p(Xf)) + Z_p(Y_p(Xf)) \\ &= X_p Y_p(Zf) - X_p Z_p(Yf) - Y_p Z_p(Xf) + Z_p Y_p(Xf). \qquad (1) \end{aligned}$$

Similarly,

$$[Y, [Z, X]]_p f = Y_p Z_p(Xf) - Y_p X_p(Zf) - Z_p X_p(Yf) + X_p Z_p(Yf) \quad (2)$$

and

$$[Z,[X,Y]]_p f = Z_p X_p(Yf) - Z_p Y_p(Xf) - X_p Y_p(Zf) + Y_p X_p(Zf). \quad (2)$$

Add (1), (2) and (3) to obtain

$$[X,[Y,Z]]_p f + [Y,[Z,X]]_p f + [Z,[X,Y]]_p f = 0$$

for any function f that is differentiable at p. Therefore

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

EXERCISE 100. Show that a connection is symmetric if and only the correspding Christoffel symbold satisfy

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

in every local chart.

SOLUTION. By Corollary 0.3 of Lecture Notes 15,

$$[X_i, X_j] = 0.$$

Thus

$$\begin{aligned} \nabla_{X_i} X_j - \nabla_{X_j} X_i &= [X_i, X_j] \\ &= 0. \end{aligned}$$

So

$$\nabla_{X_i} X_j = \nabla_{X_j} X_i.$$

But

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma^k_{ij} X_k \qquad \text{and} \qquad \nabla_{X_j} X_i = \sum_{k=1}^{n} \Gamma^k_{ji} X_k.$$

Therefore

$$\sum_{k=1}^{n} \Gamma_{ij}^{k} X_k = \sum_{k=1}^{n} \Gamma_{ji}^{k} X_k.$$

This implies that

$$\sum_{k=1}^{n} (\Gamma_{ij}^{k} - \Gamma_{ji}^{k}) X_k = 0.$$

Since $X_1, X_2, \ldots, X_k$ are linear independent, it follows that

$$\Gamma_{ij}^{k} - \Gamma_{ji}^{k} = 0 \qquad \text{for } k \in \{1, 2, \ldots, n\}.$$

Therefore

$$\Gamma_{ij}^{k} = \Gamma_{ji}^{k} \qquad \text{for } k \in \{1, 2, \ldots, n\} \text{ and } i, j \in \{1, 2, \ldots, n\}$$

in every local chart.

EXERCISE 101 (Exercise 2.7 from [Car92].). Let $S^2 \subset \mathbf{R}^3$ be the unit sphere, c an arbitrary parallel of latitude on S^2 and V_o a tangent vector to S^2 ai a point of c. Describe geometrically the parallel transport of V_o along c.

Hint: Consider the cone C tengent to S^2along c and show that the parallel transport of V_o along c is the same, whether taken relative to S^2or to C.

SOLUTION. Consider the cone C tangent to S^2 along c. Then the latitude $c(t)$ lies on the cone C. For any tange vector V_0 at the point p of $c(t)$ and q is the arbitrary point of $c(t)$. Let R_{pqt} be the rotation that has the rotation axis be the axis of the cone C and it maps p to q where $p = c(t_0)$, $q = c(t)$. Set $V(t) = R_{pqt}(V_0)$ where $R_{pqt}V_0$ is a parallel transport the vector V_0 along c. Thus the parallel transport of V_0 along c is the same, whether taken relative to S^2 or to C.\

EXERCISE 102 (Exercise 2.8 from [Car92].). Consider the upper half-plane

$$\mathbf{R}^{\mathbf{2}}_{+} = \{(x, y) \in \mathbf{R}^2; y > 0\}$$

with the metric given by $g_{11} = g_{22} \approx \frac{1}{y^2}$,$g_{12} = 0$ (metric of Lobatchevski's non-euclidean geometry).

(a) Show that the Christoffel symbols of the Riemannian connection are: $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$

(b) Let $v_o = (0, 1)$ be a tangent vector at point $(0, 1)$ or $\boldsymbol{R^{2+}}(v_o = (0, 1)$ is a unit vector on the y-axis with origin at $(0, 1))$.

Let $v(t)$ be the parallel transport of v_o along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

Hint: The field $v(t) = (a(t), b(t))$ satisfies the system (2) which defines a parallel and which, in this case, simplifies to

$$\begin{cases} \frac{da}{dt} + & \Gamma^1_{12} b = 0, \\ \frac{db}{dt} + & \Gamma^2_{11} a = 0. \end{cases}$$

Talking $a \approx \cos\theta(t)$, $b = \sin\theta(t)$ and nothing that along the given carve we have $y = 1$, we obtain from the equations above that $\frac{d\theta}{dt} = -1$. Since $v(0) = v_o$, this implies that $\theta(t) = \pi/2 - t$.

Solution. (a) We have

$$g_{ij} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

This imples that

$$g^{ij} = (g_{ij})^{-1} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

Thus

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2}(\frac{\partial}{\partial x} g_{11} + \frac{\partial}{\partial x} g_{11} - \frac{\partial}{\partial x} g_{11}) g^{11} + \frac{1}{2}(\frac{\partial}{\partial x} g_{11} + \frac{\partial}{\partial x} g_{11} - \frac{\partial}{\partial x} g_{11}) g^{21} \\ &= 0 \end{aligned}$$

since $g_{11} = \frac{1}{y^2}$, so $\frac{\partial}{\partial x}\frac{1}{y^2} = 0$ and $g^{21} = 0$. Moreover,

$$\begin{aligned} \Gamma^1_{12} &= \frac{1}{2}(\frac{\partial}{\partial x} g_{21} + \frac{\partial}{\partial y} g_{11} - \frac{\partial}{\partial x} g_{11}) g^{12} + \frac{1}{2}(\frac{\partial}{\partial x} g_{22} + \frac{\partial}{\partial y} g_{21} - \frac{\partial}{\partial y} g_{12}) g^{22} \\ &= \frac{1}{2}(\frac{\partial}{\partial x}\frac{1}{y^2}) y^2 = 0 \end{aligned}$$

since $g^{12} = 0$, $g_{21} = 0$ and $g_{22} = \frac{1}{y^2}$. Further,

$$\begin{aligned}\Gamma^1_{22} &= \frac{1}{2}(\frac{\partial}{\partial y}g_{21} + \frac{\partial}{\partial y}g_{12} - \frac{\partial}{\partial x}g_{22})g^{11} + \frac{1}{2}(\frac{\partial}{\partial y}g_{22} + \frac{\partial}{\partial y}g_{22} - \frac{\partial}{\partial y}g_{22})g^{22}\\ &= -\frac{1}{2}(\frac{-2}{y^2})\frac{1}{y^2} + \frac{1}{2}(\frac{-2}{y^2})\frac{1}{y^2} = 0.\end{aligned}$$

Hence $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$. We have

$$\begin{aligned}\Gamma^2_{11} &= \frac{1}{2}(\frac{\partial}{\partial x}g_{11} + \frac{\partial}{\partial x}g_{11} - \frac{\partial}{\partial x}g_{11})g^{12} + \frac{1}{2}(\frac{\partial}{\partial x}g_{12} + \frac{\partial}{\partial x}g_{21} - \frac{\partial}{\partial y}g_{11})g^{22}\\ &= \frac{1}{2}(\frac{\partial}{\partial x}\frac{1}{y^2} - \frac{1}{\partial y}\frac{1}{y^2})y^2\\ &= -\frac{1}{2}(-2y^{-3})y^2\\ &= \frac{y^2}{y^3} = \frac{1}{y}\end{aligned}$$

since $g^{12} = 0$. Moreover,

$$\begin{aligned}\Gamma^2_{22} &= \frac{1}{2}(\frac{\partial}{\partial y}g_{21} + \frac{\partial}{\partial y}g_{12} - \frac{\partial}{\partial y}g_{11})g^{12} + \frac{1}{2}(\frac{\partial}{\partial y}g_{22} + \frac{\partial}{\partial y}g_{22} - \frac{\partial}{\partial y}g_{22})g^{22}\\ &= \frac{1}{2}(\frac{\partial}{\partial y}\frac{1}{y^2})y^2\\ &= \frac{1}{2}(-\frac{2}{y^3})y^2\\ &= \frac{-1}{y}\end{aligned}$$

since $g^{12} = 0$. Further,

$$\begin{aligned}\Gamma^1_{12} &= \frac{1}{2}(\frac{\partial}{\partial x}g_{21} + \frac{\partial}{\partial y}g_{11} - \frac{\partial}{\partial y}g_{12})g^{11} + \frac{1}{2}(\frac{\partial}{\partial x}g_{22} + \frac{\partial}{\partial y}g_{21} - \frac{\partial}{\partial y}g_{12})g^{21}\\ &= \frac{1}{2}(\frac{\partial}{\partial y}\frac{1}{y^2})y^2\\ &= \frac{1}{2}(-\frac{2}{y^3})y^2\\ &= \frac{-1}{y}\end{aligned}$$

since $g^{21} = 0$, $g_{21} = g_{12} = 0$, $g_{11} = \frac{1}{y^2}$, $g^{11} = y^2$.

Thus $\Gamma^2_{11} = \frac{1}{y}$ and $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

(b) The field $v(t) = (a(t), b(t))$ satisfies the system

$$\frac{dv^k}{dt} + \sum_{i,j} \Gamma^k_{ij} v^j \frac{dx_j}{dt} = 0, \qquad \text{for } k = \overline{1, n}.$$

In this case, this simplies to

$$\begin{cases} \frac{da}{dt} + \Gamma^1_{12} b &= 0, \\ \frac{db}{dt} + \Gamma^2_{11} a &= 0. \end{cases} \tag{1}$$

Taking $a = \cos\theta(t)$, $b = \sin\theta(t)$. Then

$$\frac{da}{dt} = -\sin\theta(t)\frac{d\theta}{dt} \qquad \text{and} \qquad \frac{db}{dt} = \cos\theta(t)\frac{d\theta}{dt}.$$

The system (1) becomes

$$\begin{cases} -\sin\theta(t)\frac{d\theta}{dt} - \frac{1}{y}\sin\theta(t) &= 0, \\ \cos\theta(t)\frac{d\theta}{dt} + \frac{1}{y}\cos\theta(t) &= 0. \end{cases}$$

Note that $\Gamma^1_{12} = -\frac{1}{y}$, $\Gamma^2_{11} = \frac{1}{y}$ by part (a). We obtain from the system of equations that $\frac{d\theta}{dt} = -\frac{1}{y}$. But note that along the given curve, we have $y = 1$, so $\frac{d\theta}{dt} = -1$. This implies that $\theta(t) = -t + C$ for $C = \text{const}$. Subsitute $\theta(t) = -t + C$ into $v(t)$ to obtain

$$v(t) = (\cos(-t + C), \sin(-t + C)).$$

Since $v(t)$ is a parallel transport of $v_0 = (0, 1)$ along the curve $x = t$, $y = 1$, so $v(0) = v_0 = (0, 1)$. Therefore, $(\cos c, \sin c) = (0, 1)$, which implies that $C = \frac{\pi}{2}$. Hence $\theta(t) = -t + \frac{\pi}{2}$.

Exercise 103 (Exercise 2.9 from [Car92].). (Pseudo-Riemannian Metrics). A *pseudo-Riemannian* metric on a smooth manifold M is a choice, at every point $p \in M$, of a non-degenerate symmetric bilinear form $\langle , \rangle$ (not necessarily positive definite) on T_pM which varies differentiably with p. Except for the fact that $\langle , \rangle$need not be postive definite, all of the definitions that have been presendted up to now make sense for a pseudo-Riemannian

metric. For example, an affine connection on M conpatible with a pseudo-Riamannian metric on Msatisfies equation (4); if, in addition, () holds, the affine connection is said to be *symmetric*.

(a) Show that the theorem of Levi-Civita extends to pseudo-Riemannian metrics. The connection so abtained is called the *pseudo-Riemannian connection*

(b) Introduce a pseudo-Riemannian metric on $\mathbf{R}^{n+1}$ by using the quadratic form:

$$Q(x_0, ..., x_n) = -(x_o)^2 + (x_1)^2 + ... + (x_n)^2, (x_o, ..., x_n) \in \mathbf{R}^{n+1}.$$

Show that the parallel transport corresponding to the Levi_Civita connection of the metric coincides with the usual parallel transport of $\mathbf{R}^{n+1}$(this pseudo-Riemannian metric is called the Lorentz metric; for $n = 3$, it appears naturally in relativity).

SOLUTION. (a) The demonstration og the theorem of Levi-Civita dies not use positive definite and the identities which do not depend on positive definite. Therefore, for the pseudo-Riemannian connection, the theorem of Levi-Civita also holds.

(b) Let $g(x, y)$ be the pseudo-inner product with respect to the pseudo-Riemannian connection. The quadratic form $Q(x)$, $x \in \mathbf{R}^{n+1}$ satisfies $Q(x) = g(x, x)$, where

$$\begin{aligned} Q(x+y) &= g(x+y, x+y) \qquad\qquad \text{for } x, y \in \mathbf{R}^{n+1} \\ &= g(x,x) + 2g(x,y) + g(y,y) \\ & Q(x) + 2g(x,y) + Q(y). \end{aligned}$$

Therefore, $g(x, y) = \frac{1}{2}[Q(x + y) - Q(x) - Q(y)]$. For $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n)$, then

$$Q(x) = g(x, x) = -x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2,$$

$$Q(y) = g(y, y) = -y_0^2 + y_1^2 + y_2^2 + \cdots + y_n^2,$$

$$\begin{aligned} & Q(x+y) \\ =& - (x_0 + y_0)^2 + (x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + -(x_n + y_n)^2, \\ =&(-x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2) + (-y_0^2 + y_1^2 + y_2^2 + \cdots + y_n^2) + 2(-x_0y_0 + x_1y_1 + x_2y_2 + \cdots + x_ny_n) \\ =&Q(x) + Q(y) + 2(-x_0y_0 + x_1y_1 + x_2y_2 + \cdots + x_ny_n). \end{aligned}$$

This implies that

$$g(x,y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$$
$$= -x_0y_0 + x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Let $g_{ij} = g(e_i, e_j)$, $i, j = \overline{0, n+1}$ ($e_i = (0, \ldots, 0, \underbrace{1}_{i\text{th position}}, 0, \ldots, 0)$). Then

$$g_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j \geq 1, \\ -1, & \text{if } i = j = 0. \end{cases}$$

Use the formula

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \{\frac{\partial}{\partial x_i}g_{jk} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_k}g_{ij}\}g^{km}$$

to obtain

$$\Gamma_{00}^k = 0 \qquad \text{for all } k.$$

For $i, j \geq 1$, then $g_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$, so as in the Euclidean space, we have $\Gamma_{ij}^k = 0$ for all $i, j \geq 1$. thus $\Gamma_{ij}^k = 0$ for all k and all i, j. Hence

$$\frac{DV}{Dt} = \sum_k \{\frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k v^j \frac{dx_i}{dt}\}X_k = \frac{dV}{dt}.$$

Thus the parallel transport corresponding to the pseudo-Levi-Civita connection of this pseudo-metric coincides with the usual parallel transport of $\mathbf{R}^{n+1}$.

Exercise 104 (Exercise 3.1 from [Car92].). (Geodesics of a surface of revolution). Denote by (u, v) the cartesian coordinates of $\mathbf{R}^2$. Show that the function $\varphi : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ given by $\varphi(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$,

$$U = \{(u, v) \in R^2 \colon u_o < u < u_1; v_o < v < v_1\},$$

where f and g are differentiable functions, with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$, is an immersion. The image $\varphi(U)$ is the surface generated by the rotation of the curve $(f(u), g(v))$ around the axis $0z$ and is called a

surface of revolution S. The image by $\varphi(U)$ of the curves u =constant and v =constant are called *meridians* and *parallels,* respectively, of S.

(a) Show that the induced matric in the coordinates (u, v) is given by

$$g_{11} = f^2, \qquad g_{12} = 0, \qquad g_{22} = (f')^2 + (g')^2.$$

(b) Show that local equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}(\frac{du}{dt})^2 + \frac{f'f'' + g'g''}{(f'^2 + (g')^2}(\frac{dv}{dt})^2 = 0.$$

(c) Obtain the following geometric meaning of the equations above: the second equation is, except for meridians and parallels, equivalent to the fact that the "energy" $\mid \gamma'(t) \mid^2$ of a geodesic is constant along γ; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of γ with a parallel P intersecting γ at $\gamma(t)$, them

$$r \cos\beta = \text{const.},$$

where r is the radius of the parallel P (the equation above is called (*Clairaut's relation*).

(d) Use Clairaut's relation to show that a geodisic of the paraboloid

$$(f(v) = v, g(v) = v^2, 0 < v < \infty, -\varepsilon < u < 2\pi + \varepsilon),$$

which is not a meridian, intersects itself an infinite number of times (Fig. 6).

Solution. The matrix of $d\varphi$ is

$$(d\varphi(u, v)) = \begin{pmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{pmatrix}.$$

For $p = (u, v)$, then

$$d\varphi_p(\xi,\eta) = \begin{pmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$
$$= \begin{pmatrix} -f(v)\sin(u)\xi + f'(v)\cos(u)\eta \\ f(v)\cos(u)\xi + f'(v)\sin(u)\eta \\ g'(v)\eta \end{pmatrix}.$$

We have that

$$(\xi,\eta) \in \ker d\varphi_p \iff \begin{cases} -f(v)\sin(u)\xi + f'(v)\cos(u)\eta = 0 & (1) \\ f(v)\cos(u)\xi + f'(v)\sin(u)\eta = 0 & (2) \\ g'(v)\eta = 0 & (3) \end{cases}$$

* If $g'(v) \neq 0$, then it follows from (3) that $\eta = 0$. Since $(f(v)\sin u)^2 + (f(v)\cos u)^2 = f^2(v) > 0$, so it follows from (1), (2) and $\eta = 0$ that $\xi = 0$.

* If $g'(v) = 0$, then $f'(v) = 0$ (since $[f'(v)]^2 + [g'(v)]^2 \neq 0$), whence the system consisting of (1) and (2) has the determination $2f(v)f'(v) \neq 0$. Therefore it has a unique trivial solution $(0,0)$. Thus $\ker d\varphi_p = \{(0,0)\}$, so $d\varphi_p$ is an injective. Hence φ is an immersion.

(a) We have

$$X_1 = (-f(v)\sin u, f(v)\cos u, 0),$$
$$X_2 = (f'(v)\cos u), f'(v)\sin u, g'(v)).$$

Thus

$$g_{11} = \langle X_1, X_1\rangle = [f(v)]^2(\sin^2 u + \cos^2 u) = [f(v)]^2,$$
$$g_{12} = \langle X_1, X_2\rangle = 0,$$
$$g_{22} = \langle X_2, X_2\rangle = [f'(v)]^2(\sin^2 u + \cos^2 u) + [g'(v)]^2 = [f'(v)]^2 + [g'(v)]^2.$$

Therefore the induced metric in the coordinates (u,v) is

$$g_{11} = f^2, \qquad g_{12} = 0, \qquad g_{22} = (f')^2 + (g')^2.$$

(b) We have

$$(g_{ij}) = \begin{pmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{pmatrix}.$$

Therefore

$$(g^{ij}) = \begin{pmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f')^2+(g')^2} \end{pmatrix}.$$

Next, we compute Γ_{ij}^k,

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2}[\frac{\partial}{\partial x_1}g_{11} + \frac{\partial}{\partial x_1}g_{11} - \frac{\partial}{\partial x_1}g_{11}]g^{11} + \frac{1}{2}[\frac{\partial}{\partial x_1}g_{12} + \frac{\partial}{\partial x_1}g_{21} - \frac{\partial}{\partial x_2}g_{11}]g^{21} \\
&= \frac{1}{2f^2}[\frac{\partial}{\partial x_1}(f(v))^2] - \frac{1}{2}\frac{\partial}{\partial x_2}(f(v))^2\frac{1}{(f')^2 + (g')^2} = \frac{-f \cdot f'}{(f')^2 + (g')^2}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{12}^1 &= \frac{1}{2}[\frac{\partial}{\partial x_1}g_{21} + \frac{\partial}{\partial x_2}g_{11} - \frac{\partial}{\partial x_1}g_{12}]g^{11} + \frac{1}{2}[\frac{\partial}{\partial x_1}g_{22} + \frac{\partial}{\partial x_2}g_{21} - \frac{\partial}{\partial x_2}g_{12}]g^{21} \\
&= \frac{ff'}{f^2},
\end{aligned}$$

$$\begin{aligned}
\Gamma_{22}^1 &= \frac{1}{2}[\frac{\partial}{\partial x_2}g_{21} + \frac{\partial}{\partial x_2}g_{12} - \frac{\partial}{\partial x_1}g_{22}]g^{11} + \frac{1}{2}[\frac{\partial}{\partial x_2}g_{22} + \frac{\partial}{\partial x_2}g_{22} - \frac{\partial}{\partial x_2}g_{22}]g^{21} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{21}^1 &= \frac{1}{2}[\frac{\partial}{\partial x_2}g_{11} + \frac{\partial}{\partial x_1}g_{12} - \frac{\partial}{\partial x_1}g_{21}]g^{11} + \frac{1}{2}[\frac{\partial}{\partial x_2}g_{12} + \frac{\partial}{\partial x_1}g_{22} - \frac{\partial}{\partial x_2}g_{21}]g^{21} \\
&= \frac{ff'}{f^2},
\end{aligned}$$

$$\begin{aligned}
\Gamma_{11}^2 &= \frac{1}{2}[\frac{\partial}{\partial x_1}g_{11} + \frac{\partial}{\partial x_1}g_{11} - \frac{\partial}{\partial x_1}g_{11}]g^{12} + \frac{1}{2}[\frac{\partial}{\partial x_1}g_{12} + \frac{\partial}{\partial x_1}g_{21} - \frac{\partial}{\partial x_2}g_{11}]g^{22} \\
&= -\frac{1}{2}\frac{\partial}{\partial x_2}(f)^2\frac{1}{(f')^2 + (g')^2} = -\frac{ff'}{(f')^2 + (g')^2}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{12}^2 &= \frac{1}{2}[\frac{\partial}{\partial x_1}g_{21} + \frac{\partial}{\partial x_2}g_{11} - \frac{\partial}{\partial x_1}g_{12}]g^{12} + \frac{1}{2}[\frac{\partial}{\partial x_1}g_{22} + \frac{\partial}{\partial x_2}g_{21} - \frac{\partial}{\partial x_2}g_{12}]g^{22} \\
&= 0,
\end{aligned}$$

$$\Gamma_{21}^2 = 0,$$

$$\begin{aligned}\Gamma_{22}^2 &= \frac{1}{2}[\frac{\partial}{\partial x_2}g_{21} + \frac{\partial}{\partial x_2}g_{12} - \frac{\partial}{\partial x_1}g_{22}]g^{12} + \frac{1}{2}[\frac{\partial}{\partial x_2}g_{22} + \frac{\partial}{\partial x_2}g_{22} - \frac{\partial}{\partial x_2}g_{22}]g^{22}\\ &= \frac{1}{2}\frac{\partial}{\partial x_2}[(f')^2 + (g')^2]\frac{1}{(f')^2 + (g')^2} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.\end{aligned}$$

Thus the local equations of a geodesic γ are

$$\frac{d^2 x_1}{dt} + \frac{2ff'}{f^2}\frac{dx_1}{dt}\frac{dx_2}{dt} = 0,$$

$$\frac{d^2 x_2}{dt^2} + \Gamma_{11}^2(\frac{dx_1}{dt})^2 + \Gamma_{12}^2(\frac{dx_2}{dt})^2 = 0,$$

or

$$\frac{d^2 u}{dt} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0, \tag{4}$$

$$\frac{d^2 v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}(\frac{du}{dt})^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(\frac{dv}{dt})^2 = 0, \tag{5}$$

(c) Now we consider geodesics except for meridians and parallels. We have

$$\gamma'(t) = (f'v'\cos u - fu'\sin u, f'v'\sin u + fu'\cos u, g'v').$$

So

$$\begin{aligned}&|\gamma'|^2\\ &= (f'v')^2\cos^2 u + (fu')\sin^2 u - 2ff'u'v'\cos u\sin u\\ &\qquad + (f'v')^2\sin^2 u + (fu')^2\cos^2 u + 2ff'u'v'\sin u\cos u + (g'v')^2\\ &= (f'v')^2(\cos^2 u + \sin^2 u) + (fu')^2(\sin^2 u + \cos^2 u) + (g'v')^2\\ &= (f'v')^2 + (fu')^2 + (g'v')^2 \qquad\qquad \text{since } \sin^2 u + \cos^2 u = 1.\end{aligned}$$

Thus

$$|\gamma'|^2 = [(f')^2 + (g')^2](v')^2 + f^2\cdot(u')^2.$$

Taking derivative the equation above to obtain

$$\frac{d}{dt}(|\gamma'|^2) = 2(f'f'' + g'g'')(v')^3 + 2[(f')^2 + (g')^2]v'v'' + 2ff'(u')^2v' + 2f^2u'u''. \quad (6)$$

Therefore

$$\frac{1}{2}\frac{(|\gamma'|^2)'}{(f')^2 + (g')^2} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^3 + v'v'' + \frac{ff'}{(f')^2 + (g')^2}(u')^2v' + \frac{f^2u'u''}{(f')^2 + (g')^2}.$$

Substitute $v'' = \frac{ff'}{(f')^2+(g')^2}(u')^2 - \frac{f'f''+g'g''}{(f')^2+(g')^2}(v')^2$ from (5) into (6) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{(|\gamma'|^2)'}{(f')^2 + (g')^2} \\
&= \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 + v'[\frac{ff'}{(f')^2 + (g')^2}(u')^2 - \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2] + \frac{ff'}{(f')^2 + (g')^2}(u')^2v' + \frac{f^2u'u''}{(f')^2 + (g')^2} \\
&= \frac{2ff'}{(f')^2 + (g')^2}(u')^2v' + \frac{f^2}{(f')^2 + (g')^2}u'u''.
\end{aligned} \quad (7)$$

Substitute $u'' = -\frac{2ff'}{f^2}u'v'$ from (4) into (7) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{(|\gamma'|^2)'}{(f')^2 + (g')^2} &= \frac{2ff'}{(f')^2 + (g')^2}(u')^2v' - \frac{f^2}{(f')^2 + (g')^2}\frac{2ff'}{f^2}(u')^2v' \\
&= 0.
\end{aligned}$$

This implies that $(|\gamma'|^2)' = 0$, so $|\gamma'|^2 = const$.

Conversely, assume that $|\gamma'|^2 = const$. Then $(|\gamma'|^2)' = 0$, so

$$\frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^3 + v'v'' + \frac{ff'}{(f')^2 + (g')^2}(u')^2v' + \frac{f^2}{(f')^2 + (g')^2}u'u'' = 0. \quad (8)$$

Substitute $u'' = -\frac{2ff'}{f^2}u'v'$ from (4) into (8) to obtain

$$v'[\frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 + v'' + \frac{ff'}{(f')^2 + (g')^2}(u')^2 - 2\frac{f^2}{(f')^2 + (g')^2}\frac{ff'}{f^2}(u')^2] = 0.$$

Since $v' \neq 0$, so the equation above is equivalent to

$$v'' - \frac{ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 = 0.$$

Hence the equation (5) is equivalent to $|\gamma'|^2 = const$.

We have

$$\begin{aligned}(f^2u')' &= 2ff'u'v' + f^2u'' \\ &= f^2(u'' + 2\frac{ff'}{f^2}u'v') \\ &= 0 \qquad\qquad\qquad\qquad \text{by (4).}\end{aligned}$$

This implies that $f^2u' = const.$ Since

$$\cos\beta = \frac{|\langle x_u, x_u u' + x_v v'\rangle|}{|x_u|} = |fu'|$$

and $f = r$, so

$$r\cos\beta = |f^2u'| = const.$$

(d) Let p_0 be a point of the paraboloid and let P_0 be the parallel of radius r_0 passing through p_0. Let γ be a parametrized geodesic passing through p_0 and making an angle θ_0 with P_0. Since, by Clairaut's relation,

$$r\cos\theta = const. = |c|, \qquad\qquad 0 \le \theta \le \frac{\pi}{2},$$

we conclude that θ increases with r.

Therefore, if we follow in the direction of the increasing parallels, θ increases. It may happen that in some revolution surfaces γ approaches asymptotically a meridian. We shall show in a while that such is not the case with a paraboloid of revolution. That is, the geodesic γ intersects all the meridians, and therefore it makes an infinite number of turns around the paraboloid.

On the other hand, if we follow the direction of decreasing parallels, the angle θ decreases and approaches the value 0, which corresponds to a parallel of radius $|c|$ (observe that if $\theta_0 \neq 0$, $|c| < r$). Since no parallel of the paraboloid is a geodesic, the geodesic γ is actually tangent to the parallel of radius $|c|$ at the point p_1. Because 1 is a maximum for $\cos\theta$, the value of r will increase starting from p_1.We are, therefore, in the same situation as before. The geodesic will go around the paraboloid an infinite number of turns, in the direction of the increasing r's, and it will clearly intersect the other branch infinitely often.

Observe that if $\theta_0 = 0$, the initial situation is that of the point p_1.

It remains to show that when r increases, the geodesic γ meets all the meridians of the paraboloid. Observe initially that the geodesic cannot be tangent to a meridian. Otherwise, it would coincide with the meridian by the uniqueness part of Prop. 5 of Section 4.4 from [Car16]. Since the angle θ increases with r, if γ did not cut all the meridians, it would approach asymptotically a meridian, say M.

Let us assume that this is the case and let us choose a system of local coordinates for the paraboloid $z = x^2 + y^2$, given by

$$x = v\cos u, \qquad y = v\sin u, \qquad z = v^2,$$

$$0 < v < +\infty, \qquad 0 < u < 2\pi,$$

in such a way that the corresponding coordinate neighborhood contains M as $u = u_0$. By hypothesis $u \to u_0$ when $v \to \infty$. On the other hand, the equation of the geodesic y in this coordinate system is given by (cf. Eq. (6)), Example 5 from [Car16] and choose an orientation on γ such that $c > 0$)

$$u = c\int \frac{1}{v}\sqrt{\frac{1+4v^2}{v^2-c^2}}\,dv + const. > c\int \frac{dv}{d} + const.,$$

since

$$\frac{1+4v^2}{v^2-c^2} > 1.$$

It follows from the above inequality that as $v \to \infty$, u increases beyond any value, which contradicts the fact that γ approaches M asymptotically. Therefore, γ intersects all the meridians and this completes the proof of the assertion made in part (d).

Exercise 105 (Exercise 3.7 (Geodisic frame) from [Car92].). Let M be a Riemannian manifold of dimension n and let $p \in M$. Show that there exists a neighborhood $U \subset M$ of p and n vector fields $E_1, ..., E_n \in \chi(U)$, orthonormal at each point of U, such that, at p, $\nabla_{E_i} E_j(p) = 0$.

Such a family $E_i, i = 1, ..., n$, of vector fields is called a (local) *geodesic frame* at p.

SOLUTION. Let $U = B_r(p) \subset M^n$ be a normal neighborhood. For each $q \in U$, there is a nomalized geodesic γ_q joint p and q. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthogonal basis of T_pM and let $\{V_1, V_2, \ldots, V_n\}$ be their respective parallel transpots along γ_q. For each $j = \overline{1,n}$, define the field E_j by

$$E_j(q) = V_f(d(p,q))$$

where d is the Riemann distance. We have E_j is a field C^∞, because curves γ_q vary C^∞ with q in the sense that EDO's geodesics γ_q have their coefficients depending on C^∞ of q.

Now consider $\sigma_i(s)$ be the normalized geodesics such that $\sigma_i(0) = p$ and $\sigma_i'(0) = v_i = V_i(0) = E_i(p)$. We have

$$\nabla_{E_i} E_j(p) = \nabla_{E_i(p)} E_j = \nabla_{\sigma_i'(0)} E_j = \frac{D(E_j \circ \sigma_i)}{ds}|_{s=0}.$$

Since $(E_j \circ \sigma_i)(s) = V_j(d(p, \sigma_i(s))) = V_j(s)$ is a parallel field along $\gamma_{\sigma_i(s)} = \sigma_i|_{[0,s]}$, we have that

$$\begin{aligned}\nabla_{E_i} E_j(p) &= \frac{D(E_j \circ \sigma_i)}{ds}|_{s=0} \\ &= \frac{DV_j}{ds}(0) \\ &= 0.\end{aligned}$$

EXERCISE 106 (Exercise 3.8 from [Car92].). Let M be a Riemannian manifold. Let $X \in \chi(M)$ and $f \in D(M)$. Define the *divergence* of X as a function div$X$:$M \to \mathbf{R}$given by div$X(p)$ = trace of the linear mapping $Y(p) \to \nabla_Y X(p)$, $p \in M$,and the *gradient of f as a vector field grad f on M defined by*

$$\langle \mathrm{grad} f(p), v \rangle = df_p(v), p \in M, v \in T_pM.$$

(a) Let E_i, $i = 1, ..., n = \dim M$, be a geodesic frame at $p \in M$ (see Exercise Exercise 3.7). Show that:

$$\mathrm{grad} f(p) = \sum_{i=1}^{n} (E_i(f)) E_i(p),$$

$$\mathrm{div} X(p) = \sum_{i=1}^{n} E_i(f_i)(p), \qquad \text{where } X = \sum_i f_i E_i.$$

(b) Suppose that $M = \mathbf{R}^n$, with coordinates $(x_1, ..., x_n)$ and $\frac{\partial}{\partial x_i} = (0, ..., 1, ..., 0) = e_i$. show that:

$$\mathrm{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i,$$

$$\mathrm{div} X = \sum_i \frac{\partial f_i}{\partial x_i}, \qquad \text{where } X = \sum_i f_i e_i.$$

Solution. Define $f(Y) = \nabla_Y(X)$. We have

$$\begin{aligned} f(E_i) &= \nabla_{E_i} X \\ &= \sum_k (E_i(f_i) + \sum_{i,j} E_i f_j \Gamma_{ij}^k) E_k. \end{aligned} \tag{1}$$

But

$$\begin{aligned} \sum_k \Gamma_{ji}^k E_k &= \nabla_{E_j} E_i \\ &= 0 \end{aligned}$$

since $\{E_1, E_2, \ldots, E_n\}$ is a local geodesic frame at p. Since $E_1, E_2, \ldots, E_n$ are linear independent, so $\Gamma_{ji}^k = 0$, $\forall k$. This implies that $\Gamma_{ij}^k = 0$, $\forall k, i, j \in \{1, 2, \ldots, n\}$. Therefore the equation (1) becomes

$$f(E_i) = \sum_k E_i(f_k) E_k.$$

The trace of the mapping f is

$$\mathrm{trace}(f) = \sum_{i=1}^{n} E_i(f_i).$$

Thus

$$\mathrm{div}X(p) = \mathrm{trace}(f) == \sum_{i=1}^{n} E_i(f_i)(p),$$

where $X = \sum_i f_i E_i$.

Represent $\mathrm{grad}f(p)$ on the basis $\{E_1, \ldots, E_n\}$ by

$$\mathrm{grad}f(p) = \sum_{i=1}^{n} \alpha_i E_i(p).$$

Since $\{E_1, \ldots, E_n\}$ is orthonormal, so

$$\langle \mathrm{grad}f(p), E_i \rangle = \alpha_i.$$

Thus

$$\begin{aligned}\mathrm{grad}f(p) &= \sum_{i=1}^{n} \langle \mathrm{grad}f(p), E_i \rangle \, E_i \\ &= \sum_{i=1}^{n} (E_i(f)) E_i(p).\end{aligned}$$

(b) In the case $M = \mathbb{R}^n$, the $E_i = e_i$ and $E_i(f) = \langle \mathrm{grad}f, e_i \rangle = \frac{\partial f}{\partial x_i}$. Thus

$$\mathrm{grad}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i$$

and

$$\begin{aligned}\mathrm{div}f &= \sum_{i=1}^{n} (E_i(f)) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}.\end{aligned}$$

Exercise 107 (Exercise 3.10 from [Car92].). Let $f\colon [0,1] \times [0,a] \to M$ be a parametrized surface such that for all $t_o \in [0,a]$, at the point $f(0,t_o)$, $s \in [0,1]$, is a geodesic parametrized by arc length, which is orthogonal to

the curve $t \to f(0,t)$, $t \in [0,a]$, at the point $f(0,t_o)$. Prove that, for all $(s_o,t_o) \in [0,1] \times [0,a]$, the curves $s \to f(s,t_o)$, $t \to f(s_o,t)$ are orthogonal

Hint: Differentiate $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle$ with respect to s, obtaining

$$\begin{aligned} \frac{d}{ds}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle &= \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial s} \rangle \\ &= \frac{1}{2}\frac{d}{dt}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \rangle = 0, \end{aligned}$$

where we used the symmetry of the onnection and the fact that $\frac{D}{ds}\frac{\partial f}{\partial s} = 0$.

Solution. Differentiate $\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle$ with respect to s, obtaining

$$\begin{aligned} \frac{d}{ds}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle &= \left\langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle && \text{by the symmetry of the connection} \\ &= \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle && \text{since } \frac{D}{ds}\frac{\partial d}{\partial s} = 0 \end{aligned}$$

We have

$$\frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = 2\left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right\rangle \quad \text{since the inner product is symmetry.}$$

Thus

$$\left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial s} \right\rangle = \frac{1}{2}\frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle.$$

Hence

$$\begin{aligned} \frac{d}{ds}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle &= \frac{1}{2}\frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right\rangle \\ &= 0. \end{aligned}$$

Therefore

$$\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = const$$

$\forall s \in [0,1]$ and $t \in [0,a]$. In particular, we have

$$\left\langle \frac{\partial f}{\partial s}(s_0), \frac{\partial f}{\partial t}(t_0) \right\rangle = \left\langle \frac{\partial f}{\partial s}(0), \frac{\partial f}{\partial t}(t_0) \right\rangle = 0.$$

Thus the curves $s \to f(s, t_0)$, $t \to s(s_0, t)$ are orthogonal.

EXERCISE 108 (Exercise 4.4 from [Car92].). Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the curvature of M is identically zero, that is, for all $X, Y, Z \in \aleph(M)$,$R(X, Y)Z = 0$.

Hint: Consider a parametrized surface $f \colon U \subset \mathbf{R}^2 \to M$, where

$$U = \{(s,t) \in R^2; -\varepsilon < t < 1+\varepsilon, -\varepsilon < s < 1+\varepsilon, \varepsilon > 0\}$$

and $f(s, 0) = f(0, 0)$, for all s. Let $V_o \in T_{f(0,0)}(M)$ and define a field V along f by: $V(s, 0) = V_o$ and, if $t \neq 0$, $V(s,t)$ is the parallel transport of V_o along the curve $t \to f(s,t)$. Then, from Lemma 4.1,

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R(\frac{\partial f}{\partial t}\frac{\partial f}{\partial s})V.$$

Since parallel transport does not depend on the curve chosen $V(s, 1)$ is the parallel transport of $V(0, 1)$ along the curve $s \to f(s, 1)$, hence $\frac{D}{\partial s}V(s, 1) = 0$. Thus,

$$R_{f(0,1)}(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1))V(0,1) = 0.$$

Use the arbitrariness of f and V_o to conclude what is required.

SOLUTION. Consider a parametrized surface $f \colon U \subset R^2 \to M$, where

$$U = \{(s,t) \in R^2; -\epsilon < t < 1+\epsilon, -\epsilon < s < 1+\epsilon, \epsilon > 0\}$$

and $f(s, 0) = f(0, 0)$, for all s. Let $V_0 \in T_{f(0,0)}(M)$ abd define a field V along f by: $V(s, 0) = V_0$ and, if $t \neq 0$, $V(s,t)$ is the parallel tranport of V_0 along the curve $t \to f(s,t)$. Then, from Lemma 4.1,

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})V.$$

Since parallel transport does not depend on the curve chosen, $V(s, 1)$ is the parallel transport of $V(0, 1)$ along the curve $s \to f(s, 1)$, hence $\frac{D}{\partial s}V(s, 1) = 0$. Thus

$$R_{f(0,1)}(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1))V(0,1) = 0.$$

By the arbitrariness of f and V_0, and by the existence and uniqueness dependent on initial conditions theorem of the ordinary differential equations, we conclude that $R(X,Y)Z = 0$.

Exercise 109 (Exercise 4.7 from [Car92].). Prove the *2nd Bianchi Identity:*

$$\nabla R(X,Y,Z,W,T) + \nabla R(X,Y,W,T,Z) + \nabla R(X,Y,T,Z,W) = 0$$

for all $X,Y,Z,W,T \in \aleph(M)$.

Hint: Since the objects involved are all tensors, it suffices to prove the equality at a point $p \in M$. Choose a geodesic frame $\{e_i\}$ based at p (See Exercise 7 of Chap. 3). In the frame $\nabla_{e_i} e_j(p) = 0$, hence

$$\begin{aligned}\nabla R(e_i, e_j, e_k, e_{l,} e_h) &= e_h\langle R(R(e_i e_j))e_k, e_l\rangle = e_h\langle R(e_k, e_l)e_i, e_j\rangle \\ &= \langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i, e_j\rangle.\end{aligned}$$

Therefore, using the Jacobi identity for the bracket, we find

$$\begin{gathered}\nabla R(e_i, e_j, e_k, e_l, e_h) + \nabla R(e_i, e_j, e_l, e_h, e_k) \\ +\nabla R(e_i, e_j, e_h.e_k, e_l) = R(e_l, e_h, \nabla_{e_k} e_i, e_j) \\ +R(e_h, e_k, \nabla_{e_l} e_i, e_j) + R(e_k, e_l, \nabla_{e_h} e_i, e_j) = 0,\end{gathered}$$

since each one of the summands vanishes at p. The general case follows by linearity.

Solution. Choose a geodesic frame $\{e_i\}$ based at p. By Corollary 3.3 in Chapter 2 from [Car92],

$$e_h \langle R(e_k, e_l)e_i, e_j\rangle = \langle \nabla_{e_h} R(e_k, e_l)e_i, e_j\rangle + \langle R(e_k, e_l)e_i, \nabla_{e_h} e_j\rangle .$$

For the geodesic frame then $\nabla_{e_h} e_j(p) = 0$, so

$$e_h \langle R(e_k, e_l)e_i, e_j\rangle = \langle \nabla_{e_h} R(e_k, e_l)e_i, e_j\rangle . \qquad (1)$$

Since $\nabla_{e_h} e_j(p) = 0$, and R is a mutilinear mapping, so at p, we have

$$
\begin{aligned}
\nabla R(e_i,e_j,e_k,e_l,e_h) &= e_h R(e_i,e_j,e_k,e_l) - R(\nabla_{e_h} e_i, \dots e_l) - \dots - R(e_i,e_j,\dots,\nabla_{e_h} e_l)\\
&= e_h R(e_i,e_j,e_k,e_l)\\
&= e_h \langle R(e_i,e_j)e_k, e_l\rangle . \qquad (2)
\end{aligned}
$$

By Proposition 2.5 in Chapter 4 from [Car92], then

$$
\begin{aligned}
\langle R(e_i,e_j)e_k,e_l\rangle &= (e_i,e_j,e_k,e_l)\\
&= (e_k,e_l,e_i,e_j)\\
&= \langle R(e_k,e_l)e_i,e_j\rangle . \qquad (3)
\end{aligned}
$$

It follows from (1), (2) and (3) that

$$
\begin{aligned}
\nabla R(e_i,e_j,e_k,e_l,e_h) &= e_h \langle R(e_k,e_l)e_i,e_j\rangle\\
&= \left\langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i, e_j \right\rangle . \qquad (4)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\nabla R(e_i,e_j,e_l,e_h,e_k) &= \left\langle \nabla_{e_k}\nabla_{e_h}\nabla_{e_l} e_i - \nabla_{e_k}\nabla_{e_l}\nabla_{e_h} e_i + \nabla_{e_k}\nabla_{[e_l,e_h]} e_i, e_j \right\rangle , \qquad (5)\\
\nabla R(e_i,e_j,e_h,e_k,e_l) &= \left\langle \nabla_{e_l}\nabla_{e_k}\nabla_{e_h} e_i - \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i + \nabla_{e_l}\nabla_{[e_h,e_k]} e_i, e_j \right\rangle . \qquad (6)
\end{aligned}
$$

Add (4), (5) and (6) to obtain

$$
\begin{aligned}
&\nabla R(e_i,e_j,e_k,e_l,e_h) + \nabla R(e_i,e_j,e_l,e_h,e_k)\\
&\quad +\nabla R(e_i,e_j,e_h,e_k,e_l) = \langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i\\
&\qquad + \nabla_{e_k}\nabla_{e_h}\nabla_{e_l} e_i - \nabla_{e_k}\nabla_{e_l}\nabla_{e_h} e_i + \nabla_{e_k}\nabla_{[e_l,e_h]} e_i\\
&\qquad +\nabla_{e_l}\nabla_{e_k}\nabla_{e_h} e_i - \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i + \nabla_{e_l}\nabla_{[e_h,e_k]} e_i, e_j\rangle . \qquad (7)
\end{aligned}
$$

But

$$
\begin{aligned}
R(e_l,e_h,\nabla_{e_k} e_i, e_j) &= \langle R(e_l,e_h)\nabla_{e_k} e_i, e_j\rangle\\
&= \left\langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i + \nabla_{[e_l,e_h]}\nabla_{e_k} e_i, e_j \right\rangle\\
&= \left\langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i + \nabla_{[e_l,e_h]}\nabla_{e_k} e_i, e_j \right\rangle . \qquad (8)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
R(e_h,e_k,\nabla_{e_l} e_i, e_j) &= \left\langle \nabla_{e_k}\nabla_{e_h}\nabla_{e_l} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{[e_h,e_k]}\nabla_{e_l} e_i, e_j \right\rangle , \qquad (9)\\
R(e_k,e_l,\nabla_{e_h} e_i, e_j) &= \left\langle \nabla_{e_l}\nabla_{e_k}\nabla_{e_h} e_i - \nabla_{e_k}\nabla_{e_l}\nabla_{e_h} e_i + \nabla_{[e_k,e_l]}\nabla_{e_h} e_i, e_j \right\rangle . \qquad (10)
\end{aligned}
$$

We have

$$\begin{aligned}
\nabla_{e_k}\nabla_{[e_l,e_h]} - \nabla_{[e_l,e_h]}\nabla_{e_k} &= [e_k,[e_l,e_h]], &(11)\\
\nabla_{e_l}\nabla_{[e_h,e_k]} - \nabla_{[e_h,e_k]}\nabla_{e_l} &= [e_l,[e_h,e_k]], &(12)\\
\nabla_{e_h}\nabla_{[e_k,e_l]} - \nabla_{[e_k,e_l]}\nabla_{e_h} &= [e_h,[e_k,e_l]]. &(13)
\end{aligned}$$

Add (11), (12) and (13) to obtain

$$\begin{aligned}
&\nabla_{e_k}\nabla_{[e_l,e_h]} + \nabla_{e_l}\nabla_{[e_h,e_k]} + \nabla_{e_h}\nabla_{[e_k,e_l]}\\
&-(\nabla_{[e_l,e_h]}\nabla_{e_k} + \nabla_{[e_h,e_k]}\nabla_{e_l} + \nabla_{[e_k,e_l]}\nabla_{e_h}) = [e_k,[e_l,e_h]] + [e_l,[e_h,e_k]] + [e_h,[e_k,e_l]]\\
&\qquad = 0,
\end{aligned}$$

by the Jacobi identity. Therefore,

$$\nabla_{e_k}\nabla_{[e_l,e_h]} + \nabla_{e_l}\nabla_{[e_h,e_k]} + \nabla_{e_h}\nabla_{[e_k,e_l]} = \nabla_{[e_l,e_h]}\nabla_{e_k} + \nabla_{[e_h,e_k]}\nabla_{e_l} + \nabla_{[e_k,e_l]}\nabla_{e_h}. \quad (14)$$

It follows from (7), (8), (9), (10) and (14) that

$$\begin{aligned}
&\nabla R(e_i,e_j,e_k,e_l,e_h) + \nabla R(e_i,e_j,e_l,e_h,e_k)\\
&\quad +\nabla R(e_i,e_j,e_h,e_k,e_l) = R(e_l,e_h,\nabla_{e_k}e_i,e_j) + R(e_h,e_k,\nabla_{e_l}e_i,e_j)\\
&\qquad + R(e_k,e_l,\nabla_{e_h}e_i,e_j).
\end{aligned}$$

Since each one of the summands vanishes at p $(\nabla_{e_k}e_i(p) = \nabla_{e_l}e_i(p) = \nabla_{e_h}e_i(p) = 0)$, so

$$\nabla R(e_i,e_j,e_k,e_l,e_h) + \nabla R(e_i,e_j,e_l,e_h,e_k) + \nabla R(e_i,e_j,e_h,e_k,e_l) = 0.$$

The general case follows from by linearity.

Exercise 110 (Exercise 4.8 from [Car92].). *(Schur's Theorem). Let* M^n be a connected Riemannian man-ifold with $n \geq 3$. Suppose that M is isotropic, that is, for each $p \in M$,the sectional curvature $K(p,\sigma)$ does not depend on $\sigma \subset T_pM$. Prove that M has constant sectional curvature, that is, $K(p,\sigma)$ also does not depend on p.

Hint: Define a tensor $R'$of order 4 by

$$R'(W,Z,X,Y) = \langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle.$$

If $K(p,\sigma) = K$ does not depend on σ, by Lemma 3.4, $R = KR'$. Therefore, for all $U \in \aleph(M), \nabla_U R = (UK)R'$. Using the 2nd Bianchi identity (see Exercise 7):

$$\nabla R(W,Z,X,Y,U)+\nabla R(W,Z,Y,U,X)+\nabla R(W,Z,U,X,Y)=0,$$

we obtain, for all $X,Y,W,Z,U \in \aleph(M)$,

$$\begin{aligned}0 &= (UK)(\langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle) + (XK)(\langle W,Y\rangle\langle Z,U\rangle - \langle Z,Y\rangle\langle W,U\rangle)\\ &\quad + (YK)(\langle W,U\rangle\langle Z,X\rangle - \langle Z,U\rangle\langle W,X\rangle).\end{aligned}$$

Fix $p \in M$. Because $n \geq 3$, it is possible, fixing X at p, to choose Y and Z at p such that $\langle X,Y\rangle = \langle Y,Z\rangle = \langle Z,X\rangle = 0$, $\langle Z,Z\rangle = 1$. Put $U = Z$ at p. the relation above yields, for all W,

$$\langle (X,K)Y - (YK)X, W\rangle = 0.$$

Since X and Y are linearly independent at p, we conclude that $XK = 0$ for alll $X \in T_pM$. Thus $K = \text{const}$.

Solution. As in Lemma 3.4 of Chapter 4 from [Car92], we define a tensor R' of order 4 by

$$R'(W,Z,X,Y) = \langle W,X\rangle \langle Z,Y\rangle - \langle Z,X\rangle \langle W,Y\rangle .$$

If $K(p,\sigma) = K_p$ does not depend on σ, by Lemma 3.4, $R = K_pR'$. Therefore, for all $U \in \mathfrak{X}(M)$,

$$\begin{aligned}\nabla_U R &= \nabla_U K_p \cdot R'\\ &= K_p \nabla_U R' + (UK)R'\\ &= (UK)R'.\end{aligned}$$

Using the 2nd Bianchi identity (see Exercise 7):

$$\nabla R(W,Z,X,Y,U)+\nabla R(W,Z,Y,U,X)+\nabla R(W,Z,U,X,Y)=0,$$

we obtain, for all $X,Y,W,Z,U \in \mathfrak{X}(M)$,

$$\begin{aligned}&(UK_P)(\langle W,X\rangle \langle Z,Y\rangle - \langle Z,X\rangle \langle W,Y\rangle)\\ &+(XK_p)(\langle W,Y\rangle \langle Z,U\rangle - \langle Z,Y\rangle \langle W,U\rangle)\\ &+(YK_p)(\langle W,U\rangle \langle Z,X\rangle - \langle Z,U\rangle \langle W,X\rangle) = 0.\end{aligned}$$

Fix $p \in M$. Because $n \geq 3$, we can fix X at p, to choose Y and Z at p such that $\langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle = 0$, $\langle Z, Z\rangle = 1$. Put $U = Z$ at p. The relation above yields, for all W,

$$\begin{aligned}
&(XK_p)(\langle W, Y\rangle) - (YK_p)(\langle W, X\rangle) \\
&= (XK_p)(\langle Y, W\rangle) - (YK_p)(\langle X, W\rangle) \qquad \text{since the inner product is symmetric} \\
&= \langle (XK_p)Y - (YK_p)X, W\rangle \\
&= 0.
\end{aligned}$$

Thus

$$\langle (XK_p)Y - (YK_p)X, W\rangle = 0.$$

Since W is arbitrary, so take $W = (XK_p)Y - (YK_p)X$ to obtain

$$\langle (XK_p)Y - (YK_p)X, (XK_p)Y - (YK_p)X\rangle = 0.$$

This implies that

$$(XK_p)Y - (YK_p)X = 0. \qquad (1)$$

Since X and Y are independent at p (taking $X \neq 0$, $Y \neq 0$ and $\langle X, Y\rangle = 0$ above), so it follows from (1) that $XK_p = 0$ for all $X \in T_pM$. Thus $K_p = const.$

EXERCISE 111 (Exercise 4.9 from [Car92].). Prove that the scalar curvature $K(p)$ at $p \in M$ is given by

$$K(p) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(x) dS^{n-1},$$

where ω_{n-1} is the area of the sphere S^{n-1}in T_pM and dS^{n-1} is the area elements on S^{n-1}.

Hint: Use the following general argument on quadratic forms. Consider an orthonormal basis $e_1, ..., e_n$ in T_pM such that if $x \sum_{i=1}^n x_i e_i$,

$$\mathrm{Ric}_p(x) = \sum \lambda_i x_i^2, \lambda_i$$

real. Because $\mid x \mid = 1$, the vector $(x_1, ..., x_n) = v$ is a unit normal vector on S^{n-1}. Denoting $V = (\lambda_1 x_1, ..., \lambda_n x_n)$, and using Stokes Theorem, we obtain

$$\frac{1}{\omega_{n-1}}\int_{S^{n-1}}(\sum \lambda_i x_i^2)dS^{n-1} = \frac{1}{\omega_{n-1}}\int_{s^{n-1}}\langle V, v\rangle dS^{n-1}$$
$$= \frac{1}{\omega_{n-1}}\int_{B^n} \mathrm{div}V dB^n,$$

where B^n is the unit ball whose boundary is $S^{n-1} = \partial B^n$. Noting that vol $B^n/\omega_n = 1/n$, we conclude that

$$\frac{1}{\omega_{n-1}}\int_{S^{n-1}} \mathrm{Ric}_p(x)dS^{n-1} = \frac{1}{n}\mathrm{div}V = \frac{\sum \lambda_i}{n}$$
$$= \frac{\sum \mathrm{Ric}_p(e_i)}{n} = K(p).$$

Solution. We know that a quadratic form can be tranformed to a standard from (linear algebra). Therefore, there exists an orthonomrla basis $e_1, e_2, \ldots, e_n$ in $T_p(M)$ such that if $x = \sum_{i=1}^n x_i e_i$,

$$\mathrm{Ric}_p(x) = \sum_{i=1}^n \lambda_i x_i^2, \qquad \lambda_i \text{ real.}$$

Because $||x|| = 1$, the vector $\nu = (x_1, \ldots, x_n)$ is a unit normal vector on S^{n-1}. Denoting $V = (\lambda_1 x_1, \ldots, \lambda_n x_n)$, and using Stokes Theorem, we obtain

$$\frac{1}{\omega_{n-1}}\int_{S^{n-1}}(\sum_{i=1}^n \lambda_i x_i^2)\, dS^{n-1} = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \langle V, \nu\rangle\; dS^{n-1}$$
$$= \frac{1}{\omega_{n-1}}\int_{B^n} \mathrm{div}V\, dB^n,$$

where B^n is the unit ball whose boundary is $S^{n-1} = \partial B^n$.

But $\mathrm{div}V = \sum_{i=1}^n \frac{\partial V_i}{\partial x_i} = \sum_{i=1}^n \lambda_i$, $\mathrm{Ric}_p(e_i) = \lambda_i$, $\int_{B^n} dB^n = \mathrm{vol}(B^n)$, and $\frac{\mathrm{vol}(B_n)}{\omega_{n-1}} = \frac{1}{n}$, we obtain

$$\frac{1}{\omega_{n-1}}\int_{S^{n-1}} \mathrm{Ric}_p(x)\, dS^{n-1} = \frac{1}{n}\mathrm{div}V = \frac{\sum_{i=1}^n \lambda_i}{n} \qquad \text{since } \mathrm{div}V = const$$
$$= \frac{\sum_{i=1}^n \mathrm{Ric}_p(e_i)}{n} = K(p).$$

Exercise 112 (Exercise 5.1 from [Car92].). Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p\colon B_\varepsilon(0) \subset T_pM \to B_\varepsilon(p)$ is an isometry, where $B_\varepsilon(p)$ is a normal ball at p.

Solution. The mapping $\exp_p\colon B_\epsilon(0) \subset T_p(M) \to B_\epsilon(p)$ is a diffeomorphism and applying Lemma 3.5 of Chapter 3 from [Car92], we have

$$\left\langle (d\exp_p)_v v, (d\exp_p)_v w \right\rangle = \langle v, w \rangle .$$

Thus the mapping $\exp_p$ is an isometry.

Exercise 113 (Exercise 5.2 from [Car92].). Let M be a Riemannian manifold, $\gamma\colon [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curves $t \to f(t,s)$ are geodesics, such that $J(t) = \frac{\partial f}{\partial s}(t,0)$.

Hint: Choose a curve $\lambda(s)$, $s \in (-\varepsilon, \varepsilon)$ in M such that $\lambda(0) = \gamma(0)$, $\lambda'(0) = J(0)$. Along λ choose a vector field $W(s)$ with $W(0) = \gamma'(0)$, $\frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$. Define $f(s,t) = \exp_{\lambda(s)} tW(s)$ and verify that $\frac{\partial f}{\partial s}(0,0) = \frac{d\lambda}{ds}(0) = J(0)$ and

$$\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) = \frac{DW}{ds}(0) = \frac{DJ}{dt}(0).$$

Solution. We choose a curve $\lambda(s)$, $s \in (-\epsilon, \epsilon)$ in M such that $\lambda(0) = \gamma(0)$, $\lambda'(0) = J(0)$. Along λ choose a vector field $W(s)$ with $W(0) = \gamma'(0)$, $\frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$.

Define $f(s,t) = \exp_{\lambda(s)} tW(s)$. We have

$$\begin{aligned}
\frac{\partial f}{\partial s}(0,0) &= J(0) \\
&= \lambda'(0) \\
&= \frac{d\lambda}{ds}(0), \\
\frac{D}{dt}\frac{\partial f}{\partial s}(0,0) &= \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) && \text{by Lemma 3.4} \\
&= \frac{DW}{ds}(0) \\
&= \frac{DJ}{dt}(0).
\end{aligned}$$

Thus

$$J(t) = \frac{\partial f}{\partial s}(t, 0) \qquad\qquad \text{by Proposition 2.4.}$$

EXERCISE 114 (Exercise 5.3 from [Car92].). Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C(p)$ is empty.

Hint: Assume the existence of a non-trivial Jacobi field along the geodesic $\gamma\colon [0, a] \to M$, with $\gamma(0) = p$, $J(0) = J(a) = 0$. Use the Jacobi equation to show that $\frac{d}{dt}\langle \frac{DJ}{dt}, J\rangle \geq 0$. Conclude that $\langle \frac{DJ}{dt}, J\rangle \equiv 0$. Since $\frac{d}{dt}\langle J, J\rangle = 2\langle \frac{DJ}{dt}, J\rangle \equiv 0$, we have $|| J ||^2 =$ const. $= 0$, a contradiction

SOLUTION. Asume the existence of a non-trivial Jacobi field along the geodesic $\gamma\colon [0, a] \to M$, with $\gamma(0) = p$, $J(0) = J(a) = 0$. We have

$$\begin{aligned}\frac{d}{dt}\left\langle \frac{DJ}{dt}, J\right\rangle &= \left\langle \frac{D^2 J}{dt^2}, J\right\rangle + \left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle \\ &= \langle -R(\gamma'(t)), J(t))\gamma'(t), J\rangle + \left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle \quad \text{by the Jacobi equation.}\end{aligned}$$

Since non-position section curvature and $\left\langle \frac{DJ}{dt}, \frac{DJ}{dt}\right\rangle \geq 0$, so the equation above implies

$$\frac{d}{dt}\left\langle \frac{DJ}{dt}, J\right\rangle \geq 0.$$

Define

$$g(t) = \left\langle \frac{DJ}{dt}(t), J(t)\right\rangle.$$

Since $\frac{dg}{dt} = \frac{d}{dt}\left\langle \frac{DJ}{dt}, J\right\rangle \geq 0$, so $g(t)$ is increasing, therefore

$$0 = g(0) \leq g(t) \leq g(a) = \left\langle \frac{DJ}{dt}(a), J(a)\right\rangle = 0.$$

Thus

$$g(t) \equiv 0 \qquad\qquad \text{for all } t \in [0, a].$$

or

$$\left\langle \frac{DJ}{dt}, J \right\rangle \equiv 0.$$

Since $\frac{d}{dt}\langle J, J\rangle = 2\left\langle \frac{DJ}{dt}, J\right\rangle = 0$, so $||J||^2 = const = 0$ (since $J(0) = 0$). This contradicts to non-trivial Jacobi field.

Exercise 115 (Exercise 5.6 from [Car92].). Let M be a Riemannian manifold of dimension two (in this case we say that M is a surface). Let $B_\partial(p)$ be a normail ball around the point $p \in M$ and consider the parametrized surface

$$f(\rho, \theta) = \exp_p \rho v(\theta), \qquad 0 < \rho < \delta, -\pi < \theta < \pi,$$

where $v(\theta)$ is a circle of radius δ in T_pM parametrized by the central angle θ.

(a) Show that (ρ, θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(0))0 < \rho < \delta$. Such coordinates are called *polar coordinates* at p.

(b) Show that the coeffcients g_{ij} if the Riemannian metrix in these coordinates are:

$$g_{12} = 0, \qquad g_{11} = \left| \frac{\partial f}{\partial \rho} \right|^2 = \left| v(\theta) \right|^2 = 1, \qquad g_{22} \approx \left| \frac{\partial f}{\partial \theta} \right|^2.$$

(c) Show that, along the geodesic $f(\rho, 0)$, we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho),$$

where $\lim\limits_{\rho\to 0} \frac{R(\rho)}{\rho} = 0$ and $K(p)$ is the sectional curvature of M at p.

d) Prove that

$$\lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p).$$

This last expression is the value of the Gaussian curvature of M at p given in polar coordinates (Cf., for example [Car76] p. 288). This fact from the theory of surfaces, and (d) shows that, in dimension two, the sectional curvature coincides with the Gaussian curvature. In the next chapter, we shall give a more direct proof of this fact.

SOLUTION. (a) Let $\exp_p(l) = L$, where L is the ray $\exp_p(-\rho v(\theta))$, $0 < \rho < \delta$. Let $x\colon U\backslash l \to B_\delta(p)\backslash L$ be a system of geodesic polar coordinates (ρ, θ). Every point in $B_\delta(\rho)\backslash L$ is defined uniquely by a some pair (ρ, θ) and converse since $\exp_p$ is an isomorphism. Thus (p, θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(\theta))$, $0 < \rho < \delta$.

(b) Since $g_{12} = \left\langle \frac{\partial x}{\partial \rho}, \frac{\partial x}{\partial \theta} \right\rangle$, so

$$(g_{12})_p = \left\langle \frac{\partial^2 x}{\partial \rho^2}, \frac{\partial x}{\partial \theta} \right\rangle + \left\langle \frac{\partial x}{\partial \rho}, \frac{\partial^2 x}{\partial \theta \partial \rho} \right\rangle.$$

Since $\theta =$ const, is a geodesic, we have

$$\begin{aligned}(g_{12})_p &= \left\langle \frac{\partial x}{\partial \rho}, \frac{\partial}{\partial \theta}(\frac{\partial x}{\partial \rho}) \right\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial \theta} \left\langle \frac{\partial x}{\partial \rho}, \frac{\partial x}{\partial \rho} \right\rangle = 0.\end{aligned}$$

For each $q \in B_\delta(p)$, we denote $\alpha(\sigma)$ as the geodesic circle that passes through q, where $\sigma \in (-\pi, \pi)$. (Note that if $q = p$ then $\alpha(\sigma)$ is the constant curve $\alpha(\sigma) = p$.) We denote $\gamma(s)$, where s is the arc length of γ, as the radical geodesic that passes through q. With this notation we may write

$$g_{12}(\rho, \theta) = \left\langle \frac{d\alpha}{d\sigma}, \frac{d\gamma}{ds} \right\rangle.$$

The coefficient $g_{12}(\rho, \theta)$ does not defined at p. However, if we fix the radical geodesic $\theta =$ const, the second number of the above equation is defined for every point of this geodesic. Since at p, $\alpha(\sigma) = p$, so $\frac{d\alpha}{d\sigma} = 0$, we obtain

$$\lim_{\rho \to 0} g_{12} = \lim_{\rho \to 0} \left\langle \frac{d\alpha}{d\sigma}, \frac{d\gamma}{ds} \right\rangle = 0.$$

Since $(g_{12})_p = 0$, so g_{12} does not depend on p and the fact above implies that $g_{12} = 0$.

By the definition of the exponential map, ρ measures the arc length along the curve $\theta =$ const. This implies that

$$g_{11} = |\frac{\partial f}{\partial \rho}|^2 = |v(\theta)|^2 = 1$$

and we also have

$$g_{22} = \left\langle \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \theta} \right\rangle = |\frac{\partial f}{\partial \theta}|^2.$$

(c) By Corollary 2.10 of Chapter 5 from [Car92], then

$$|J(\rho)| = \rho - \frac{1}{6}K(p)\rho^3 + \overline{R}(\rho), \qquad \text{where } \lim_{\rho \to 0} \frac{\overline{R}(\rho)}{\rho^3} = 0.$$

Since $\sqrt{g_{22}} = |\frac{\partial f}{\partial \theta}| = |J(p)|$, so $\sqrt{g_{22}} = \rho - \frac{1}{6}K(p)\rho^3 + \overline{R}(\rho)$. Taking derivative twice with respect to ρ to obtain

$$(\sqrt{g_{12}})_{\rho\rho} = -K(p)\rho + (\overline{R})_{\rho\rho}.$$

Put $R = (\overline{R})_{\rho\rho}$, then

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho). \qquad (1)$$

Using l'Hopital rule to obtain

$$0 = \lim_{\rho \to 0} \frac{\overline{R}}{\rho^3} = \lim_{\rho \to 0} \frac{(\overline{R})_{\rho\rho}}{6\rho}.$$

Hence

$$\lim_{\rho \to 0} \frac{(\overline{R})_{\rho\rho}}{\rho} = 0.$$

Thus

$$\lim_{\rho \to 0} \frac{R(\rho)}{\rho} = 0. \qquad (2)$$

Equations (1) and (2) satisfy the requires of the problem.

(d) Firstly, we shall prove that

$$\lim_{\rho \to 0} \frac{\sqrt{g_{22}}}{\rho} = 1.$$

Indeed,

$$\lim_{\rho\to 0}\frac{\sqrt{g_{22}}}{\rho} = \lim_{\rho\to 0}\frac{|J(p)|}{\rho}$$
$$= \lim_{\rho\to 0}\frac{\rho - \frac{1}{6}K(p)\rho^3 + \overline{R}(\rho)}{\rho} = 1$$

since $\lim_{\rho\to 0}\frac{\overline{R}(\rho)}{\rho} = \lim_{\rho\to 0}\rho^2\frac{\overline{R}(\rho)}{\rho^3} = 0$. Next, we have

$$\begin{aligned}
\lim_{\rho\to 0}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} &= \lim_{\rho\to 0}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\rho}\cdot\lim_{\rho\to 0}\frac{\rho}{\sqrt{g_{22}}} \\
&= \lim_{\rho\to 0}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\rho} \qquad\qquad \text{since } \lim_{\rho\to 0}\frac{\rho}{\sqrt{g_{22}}} = 1 \\
&= \lim_{\rho\to 0}\frac{-K(p)\rho + R(\rho)}{\rho} \\
&= -K(p)
\end{aligned}$$

since $\lim_{\rho\to 0}\frac{R(\rho)}{\rho} = 0$ and $\sqrt{g_{22}} = -K(p)\rho + R(\rho)$.
Therefore

$$\lim_{\rho\to 0}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p).$$

Exercise 116 (Exercise 5.7 from [Car92].). Let M be a Riemannian manifold of dimension two. Let $p\in M$ and let $V\subset T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0)\subset V$ be a circle of radius r centered at the origin, and let L_r be the length of the curve $\exp_p(S_r)$ in M. Prove that the sectional curvatire at $p\in M$ is given by

$$K(p) = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L_r}{r^3}.$$

Hint: Use Exercise 6.

Solution. (a) By Exercise 6, we have

$$\sqrt{g_{22}} = \rho - \frac{1}{6}K(p)\rho^3 + R(\rho).$$

We also have

$$L_r = \lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{\pi-\epsilon}\sqrt{g_{22}}\,d\theta = 2\pi r - \frac{\pi}{3}r^3K(\rho) + R_1$$

where $\lim_{r\to 0}\frac{R_1}{r^3} = 0$. This implies that

$$\begin{aligned} K(p) &= \lim_{r\to 0}\frac{3}{\pi}\Big(\frac{2\pi r - L_r + R_1}{r^3}\Big) \\ &= \lim_{r\to 0}\frac{3}{\pi}\cdot\frac{2\pi r - L_r}{r^3} \qquad\qquad \text{since } \lim_{r\to 0}\frac{R_1}{r^3} = 0. \end{aligned}$$

EXERCISE 117 (Exercise 6.1 from [Car92].). Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product matric. Let $\nabla^{\mathbf{1}}$be the Riemannian connection of M_1 and let $\nabla^{\mathbf{2}}$ be the Riemannian connection of M_2

(a) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by $\nabla_{Y_1+Y_2}(X_1+X_2) = \nabla^1_{Y_1}X_1 + \nabla^2_{Y_2}X_2, X_1, Y_1 \in X(M_1), X_2, Y_2 \in X(M_2)$.

(b) For every $p \in M_1$, the set $(M_2)_p = \{(p,q) \in M_1 \times M_2; q \in M_2\}$ is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.

(c) Let $\sigma(x,y) \subset T_{(p,q)}(M_1 \times M_2)$ be a plane such that $x \in T_pM_1$ and $y \in T_qM_2$. Show that $K(\sigma) = 0$.

SOLUTION. (a) Let $p \in M_1$, $q \in M_2$. Consider the carmonical maps $M_1 \to M_1 \times \{q\}$ and $M_2 \to \{p\} \times M_2$ be immersions of M_1 and M_2 into $M_1 \times M_2$. For $X_1, Y_1 \in M_1 \times \{q\}$, $X_2, Y_2 \in \{p\} \times M_2$,

$$\nabla^1_{Y_1}X_1 = (\nabla_{\overline{Y}}\overline{X})^{\tan M_1},$$

where $\overline{X}$, $\overline{Y}$ are extensions of X_1 and Y_1. We get a similar formula for $\nabla^2_{Y_2}X_2$. Define $\overline{X}$, $\overline{Y}$ by

$$\begin{aligned} \overline{X}(a,b) &= X_1(a,q) + X_2(p,b), \\ \overline{Y}(a,b) &= Y_1(a,q) + Y_2(p,b), \end{aligned}$$

for all $(a,b) \in M_1 \times M_2$. This gives

$$\begin{aligned} \nabla^1_{Y_1}X_1 + \nabla^2_{Y_2}X_2 &= (\nabla_{\overline{Y}}\overline{X})^{\tan M_1} + (\nabla_{\overline{Y}}\overline{X})^{\tan M_2} \\ &= \nabla_{\overline{Y}}\overline{X}, \end{aligned}$$

as required.

(b) Since M_2 is natural diffeomorphic to $(M_2)_p$ and by part (a), so

$$\overline{\nabla}^2_{\overline{Y}}\overline{X} = \nabla^2_Y X \qquad\qquad \text{for } Y, X \in M_2,$$

where $\overline{\nabla}^2_{\overline{Y}}\overline{X} = \nabla_Y X$. Thus $B(X,Y) = 0$ for all $X, Y \in (M_2)_p$. Therefore $B(x,x) = 0$ for all $x \in (M_2)_p$. For every $y \in M_1 \times M_2$, $y = (y_1, y_2)$, where $y_1 \in M_1$, $y_2 \in M_2$. So $y \in (M_2)_{y_1}$. This implies that $B(y,y) = 0$ for all $y \in M_1 \times M_2$. Hence $(M_2)_p$ is totally geodesic on $M_1 \times M_2$.

(c) Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$ be basis in T_pM_1 and T_qM_2, respectively, where $X_i = \frac{\partial}{\partial x_i}$, $Y_j = \frac{\partial}{\partial y_j}$. Whence $((X_i, Y_j))$ is a basis in $T_{(p,q)}(M_1 \times M_2)$. Since $\nabla^1_{X_i} X_j = 0$ and $\nabla^2_{Y_i} Y_j = 0$ for all i, j, so $\nabla_{(X_i,Y_j)}(X_k Y_l) = 0$ and $[(X_i, Y_j), (X_k, Y_l)] = 0$ for all i, j, k, l. Thus $(x, y, x, y) = 0$ for all $x, y \in \sigma$ (since (x, y, u, v) is multilinear). Therefore $K(\sigma) = 0$.

Exercise 118 (Exercise 6.2 from [Car92].). Show that $\mathbf{x} : \mathbf{R^2} \to \mathbf{R^4}$given by

$$\mathbf{x}(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \qquad\qquad (\theta, \varphi) \in \mathbf{R^2}$$

is an immersion of $\mathbf{R^2}$into the unit sphere $S^3(1) \subset \mathbf{R^4}$, whose image $\mathbf{x}(\mathbf{R^2})$ is a torus $T^2$with sectional curvature zero in the induced matric

Solution. We have

$$dx = \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\theta & 0 \\ -\frac{1}{\sqrt{2}}\cos\theta & 0 \\ 0 & -\frac{1}{\sqrt{2}}\sin\theta \\ 0 & -\frac{1}{\sqrt{2}}\cos\theta \end{pmatrix}$$

which is certainly of rank 2, so x is an immersion. Moreover,

$$\begin{aligned} ||x(\theta, \varphi)|| &= \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi) \\ &= 1, \end{aligned}$$

so the image of x is contained in $S^3(1)$. Since $(\cos\theta, \sin\theta)$ and $(\cos\varphi, \sin\varphi)$ parametrize the unit circle, so the image of x is $S^1 \times S^1 = T^2$.

Define $X_i = \frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial \theta} = \frac{1}{\sqrt{2}}(-\sin\theta X_1 + \cos\theta X_2)$ and $\frac{\partial}{\partial \varphi} = \frac{1}{\sqrt{2}}(-\sin\varphi, X_3 + \cos\varphi, X_4)$. We note that $\overline{K}(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}) = 0$ since all sectional curvature of R^4 are zero. Hence, by Gauss Theorem,

$$
\begin{aligned}
K(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}) &= K(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}) - \overline{K}(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}) \\
&= \left\langle B(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}), B(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}) \right\rangle - |B(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi})|^2 \\
&= \left\langle \overline{\nabla}_{\overline{\partial/\partial\theta}} \overline{\frac{\partial}{\partial\theta}} - \nabla_{\partial/\partial\theta} \frac{\partial}{\partial\theta}, \overline{\nabla}_{\overline{\partial/\partial\varphi}} \overline{\frac{\partial}{\partial\varphi}} - \nabla_{\partial/\partial\varphi} \frac{\partial}{\partial\varphi} \right\rangle - |\overline{\nabla}_{\overline{\partial/\partial\theta}} \overline{\frac{\partial}{\partial\varphi}} - \nabla_{\partial/\partial\theta} \frac{\partial}{\partial\varphi}|^2 \\
&= \left\langle \overline{\nabla}_{\overline{\partial/\partial\theta}} \overline{\frac{\partial}{\partial\theta}}, \overline{\nabla}_{\overline{\partial/\partial\varphi}} \overline{\frac{\partial}{\partial\varphi}} \right\rangle - |\overline{\nabla}_{\overline{\partial/\partial\theta}} \overline{\frac{\partial}{\partial\varphi}}|^2
\end{aligned}
$$

since $\nabla_{\partial/\partial\theta} \frac{\partial}{\partial\theta} = 0$, $\nabla_{\partial/\partial\varphi} \frac{\partial}{\partial\varphi} = 0$ and $\nabla_{\partial/\partial\theta} \frac{\partial}{\partial\varphi} = 0$ (because θ and φ give coordinates on $\mathbb{R}^2$). Consider the vector field V where $V(x_1, x_2, x_3, x_4) = \frac{1}{\sqrt{2}}(-x_2, x_1, -x_4, x_3)$. Then

$$
\begin{aligned}
\overline{\nabla}_{\overline{\partial/\partial\theta}} \overline{\frac{\partial}{\partial\theta}} &= \overline{\nabla}_V V \\
&= \frac{1}{2} \nabla_{-x_2 X_1 + x_1 X_2}(-x_2 X_1 + x_1 X_2) \\
&= \frac{1}{2}[-x_2 \nabla_{X_1}(-x_2 X_1 + x_1 X_2) + x_1 \nabla_{X_2}(-x_2 X_1 + x_1 X_2)] \\
&= \frac{1}{2}[-x_2 X_1(x_1) X_2 + x_1 X_2(-x_2) X_1] \\
&= \frac{1}{2}[-x_1 X_1 - x_2 X_2]
\end{aligned}
$$

since all the other terms in the expression of the third line are zero. A similar argument shows that

$$
\overline{\nabla}_{\overline{\partial/\partial\varphi}} \overline{\frac{\partial}{\partial\varphi}} = \frac{1}{2}[-x_3 X_3 - x_4 X_4]
$$

and $\overline{\nabla}_{\overline{\partial/\partial\theta}} \overline{\frac{\partial}{\partial\varphi}} = 0$. Therefore

$$
\begin{aligned}
K(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\varphi}) &= \left\langle \overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\theta}}, \overline{\nabla}_{\overline{\partial/\partial\varphi}}\overline{\frac{\partial}{\partial\varphi}} \right\rangle - |\overline{\nabla}_{\overline{\partial/\partial\theta}}\overline{\frac{\partial}{\partial\varphi}}|^2 \\
&= \frac{1}{4}(-x_1X_1 - x_2X_2 - x_3X_3 - x_4X_4) - 0 \\
&= 0.
\end{aligned}
$$

Exercise 119 (Exercise 6.3 from [Car92].). Let M be a Riemannian manifold and let $N \subset K \subset M$ be submanifolds of M. Suppose that N is totally geodesic in K and that K is totally geodesic in M. Prove that N is totally geodesic in M.

Solution. For every γ is geodesic on N, then γ is also geodesic on K because N is totally geodesic in K. Since K is totally geodesic in M, so γ is also geodesic on M. Thus every geodesic on K is also geodesic on M. Therefore N is totally geodesic on M.

Exercise 120 (Exercise 6.6 from [Car92].). Let G be a Lie group with a bi-invariant metric. Let H be a Lie group and let $h\colon H \to G$ be an immersion that is also a homomorphism of groups (that is, H is a Lie subgroup of G). Show that h is a totally geodesic immersion.

Solution. Since the metric H inherits from G is bi-invariant on H, we know that the geodesics through the identity in H are one-parameter subgroups of H. But a one-parameter subgroup of H is also a one-parameter subgroup of G. Since the one-parameter subgroups of G are just geodesics through the identiy, this implies that the geodesics through the identity in H are geodesics in G, which demonstrates that H is a geodesics immersion at the identity. Since lift-translation is an isometry of H and istrometries take geodesics to geodesics, this in turn means that H is a geodesic immersion at every $p \in H$, so H is totally geodesic immerson.

The metric inherited from G on H is simply

$$\langle u, v\rangle_H = \langle dh_g(u), dh_g(v)\rangle_G$$

where $u, v \in T_gH$. We shall prove that the metric H inherited from G is bi-invariant on H. Suppose $u, v \in T_gH$. Then

$$
\begin{aligned}
\langle u, v\rangle_H &= \langle dh_g(u), dh_g(v)\rangle_G \\
&= \big\langle (dL_{h(g')})_g \circ (dh_g)u, (dL_{h(g')})_g \circ (dh_g)v\big\rangle_G \\
&= \big\langle d(L_{h(g')} \circ h)_g u, d(L_{h(g')} \circ h)_g v\big\rangle_G \\
&= \big\langle d(h \circ h^{-1}|_{g'} \circ L_{h(g')} \circ h)_g u, d(h \circ h^{-1}|_{g'} \circ L_{h(g')} \circ h)_g v\big\rangle_G \\
&= \big\langle d(h \circ L_{g'})_g u, d(h \circ L_{g'})_g v\big\rangle_G \\
&= \big\langle (dh)_{g'} \circ (dL_{g'})_g u, (dh)_{g'} \circ (dL_{g'})_g v\big\rangle_G \\
&= \big\langle (dL_{g'})_g u, (dL_{g'})_g v\big\rangle_H .
\end{aligned}
$$

A similar calculation shows that this inherited metric is righ-invariant, so it is a bi-invariant metric.

Exercise 121 (Exercise 6.8 (*The Clifford torus*) from [Car92].). . Consider the immersion $x\colon \mathbf{R^2} \to \mathbf{R^4}$given in Exercise 2.

(a) Show that the vectors

$$e_1 = (-\sin\theta, \cos\theta, 0, 0), \qquad\qquad e_2 = (0, 0, -\sin\varphi, \cos\varphi)$$

form an orthonormal basis of the tangent space, and that the vectors $n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$, $n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$ form an orthonor-mal basis of the normal space.

b) Use the fact that

$$\langle S_{n_k}(e_i), e_j\rangle = -\langle \overline{\nabla}_{e_i} n_k, e_j\rangle = \langle \overline{\nabla}_{e_i} e_j, n_k\rangle,$$

where $\overline{\nabla}$ is the covariant derivative (that os, the usual derivative) of $\mathbf{R^4}$, and $i, j, k = 1, 2$, to establish that the matrices of $S_{n_1}$and $S_{n_2}$with respect to the basic $\{e_1, e_2\}$ are

$$S_{n_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(c) From Exercise 2, $\mathbf{x}$ is an immersion of the torus T^2 into $S^3(1)$ (the Clifford torus). Show that $\mathbf{x}$ is a minimal immersion.

SOLUTION. (a) By the solution of Exercise 3, we have $\operatorname{rank}(dx) = 2$. So the dimension of the tangent space are 2. We have

$$\begin{aligned}
\langle e_1, e_2\rangle &= -\sin\theta \cdot 0 + \cos\theta \cdot 0 + 0 \cdot (-\sin\theta) + 0\cos\theta = 0,\\
\langle e_1, e_1\rangle &= (-\sin\theta)^2 \cdot 0 + (\cos\theta)^2 = 1,\\
\langle e_2, e_2\rangle &= 0^2 + 0^2 + (-\sin\theta)^2 + (\cos\theta)^2 = 1.
\end{aligned}$$

Therefore e_1 and e_2 form an orthogonal basis of the tangent space. Moreover, for all x in the tangent space, $x = \alpha_1 e_1 + \alpha_2 e_2$ where α_1 and α_2 belong to $\mathbb{R}$ and

$$\begin{aligned}
\langle n_1, x\rangle &= \alpha_1 \langle n_1, e_1\rangle + \alpha_2 \langle n_1, e_2\rangle\\
&= \frac{\alpha_1}{\sqrt{2}}(-\sin\theta\cos\theta + \cos\theta\sin\theta) + \frac{\alpha_2}{\sqrt{2}}(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi),\\
\langle n_2, x\rangle &= \alpha_1 \langle n_2, e_1\rangle + \alpha_2 \langle n_2, e_2\rangle\\
&= \frac{\alpha_1}{\sqrt{2}}(\sin\theta\cos\theta - \cos\theta\sin\theta) + \frac{\alpha_2}{\sqrt{2}}(-\sin\varphi\cos\varphi + \cos\varphi\sin\varphi) = 0.
\end{aligned}$$

So n_1 and n_2 are orthogonal to the tangent space, that is, n_1 and n_2 belong to the orthogonal complement of the tangent space, and

$$\begin{aligned}
\langle n_1, n_2\rangle &= \frac{1}{2}(-\cos^2\theta - \sin^2\theta + \cos^2\varphi + \sin^2\varphi) = 0,\\
\langle n_1, n_1\rangle &= \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi) = 1,\\
\langle n_2, n_2\rangle &= \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi) = 1.
\end{aligned}$$

Therefore n_1 and n_2 form an orthogonal basis of the normal space.

(b) We have

$$e_1 = (-\sin\theta, \cos\theta, 0, 0) = \sqrt{2} + \frac{1}{\sqrt{2}}(-\sin\theta, \cos\theta, 0, 0) = \sqrt{2}\frac{\partial}{\partial\theta}.$$

Similarly,

$$e_2 = \sqrt{2}\frac{\partial}{\partial\varphi}.$$

Since

$$\langle S_{n_k}(e_i), e_j \rangle = -\left\langle \overline{\nabla}_{e_i} n_k, e_j \right\rangle = \left\langle \overline{\nabla}_{e_i} e_j, n_k \right\rangle,$$

so

$$\begin{aligned}
\langle S_{n_1}(e_1), e_1 \rangle = \left\langle \overline{\nabla}_{e_1} e_1, n_1 \right\rangle &= \left\langle \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\theta}} \sqrt{2}\frac{\partial}{\partial\theta}, n_1 \right\rangle \\
&= \left\langle -\cos\theta X_1 - \sin\theta X_2, \frac{1}{\sqrt{2}}(\cos\theta X_1 + \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4) \right\rangle \\
&= \frac{1}{\sqrt{2}}(-\cos^2\theta - \sin^2\theta) = \frac{-1}{\sqrt{2}},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\langle S_{n_1}(e_1), e_2 \rangle &= \left\langle \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\theta}} \sqrt{2}\frac{\partial}{\partial\varphi}, n_1 \right\rangle \\
&= 0 \qquad\qquad \text{since } \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\theta}} \sqrt{2}\frac{\partial}{\partial\varphi} = (0,0,0,0)
\end{aligned}$$

and

$$\langle S_{n_1}(e_2), e_1 \rangle = 0,$$

and

$$\begin{aligned}
\langle S_{n_1}(e_2), e_2 \rangle &= \left\langle \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\varphi}} \sqrt{2}\frac{\partial}{\partial\varphi}, n_1 \right\rangle \\
&= \left\langle -\cos\varphi X_3 - \sin\varphi X_4, \frac{1}{\sqrt{2}}(\cos\theta X_1 + \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4) \right\rangle \\
&= \frac{1}{\sqrt{2}}(-\cos^2\varphi - \sin^2\varphi) = \frac{-1}{\sqrt{2}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\langle S_{n_2}(e_1), e_1 \rangle = \left\langle \overline{\nabla}_{e_1} e_1, n_2 \right\rangle &= \left\langle \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\theta}} \sqrt{2}\frac{\partial}{\partial\theta}, n_2 \right\rangle \\
&= \left\langle -\cos\theta X_1 - \sin\theta X_2, \frac{1}{\sqrt{2}}(-\cos\theta X_1 - \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4) \right\rangle \\
&= \frac{1}{\sqrt{2}}(\cos^2\theta \sin^2\theta) = \frac{-1}{\sqrt{2}},
\end{aligned}$$

and similarly,

$$\langle S_{n_2}(e_1), e_2\rangle = \left\langle \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\theta}} \sqrt{2}\frac{\partial}{\partial\varphi}, n_2 \right\rangle$$
$$= 0 \qquad\qquad \text{since } \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\theta}} \sqrt{2}\frac{\partial}{\partial\varphi} = (0,0,0,0)$$

and

$$\langle S_{n_2}(e_2), e_1\rangle = 0,$$

and

$$\begin{aligned}\langle S_{n_2}(e_2), e_2\rangle &= \left\langle \overline{\nabla}_{\sqrt{2}\frac{\partial}{\partial\varphi}} \sqrt{2}\frac{\partial}{\partial\varphi}, n_2 \right\rangle \\ &= \left\langle -\cos\varphi X_3 - \sin\varphi X_4, \frac{1}{\sqrt{2}}(-\cos\theta X_1 - \sin\theta X_2 + \cos\varphi X_3 + \sin\varphi X_4) \right\rangle \\ &= \frac{1}{\sqrt{2}}(-\cos^2\varphi - \sin^2\varphi) = \frac{-1}{\sqrt{2}}.\end{aligned}$$

The metrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are

$$S_{n_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$S_{n_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(c) Since

$$\left\| \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) \right\|^2 = \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\varphi + \sin^2\varphi) = 1,$$

so n_1 is thought of as a point which lies on the sphere $S^3(1)$, so n_1 is thought of as a vector which is orthogonal to $S^3(1)$. Therefore, the subspace of T_pS^3 is normal to T_pT^2, where T^2 is the image of x spanned by n_2. Therefore the condition of minimal that trace $S_{n_2} = 0$, which is clearly does, given the computation in part (b) above ($S_{n_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$). Thus x is a minimal immersion.

Exercise 122 (Exercise 6.10 from [Car92].). Let $f\colon M^n \to \overline{M}^{n+k}$ be an isometric immersion and let $S_\eta\colon TM \to TM$ be the operator associated to the second fundamental form of f along the normal field η. Consider

S_η as a tensor of order 2 given by $S_\eta(X,Y) = \langle S_\eta(X), Y\rangle, X, Y \in X(M)$. Observe that saying the operator S_η is selfadjoint is equivalent to saying that the tensor $S_\eta$os symmetric, that is, $S_\eta(X,Y) = S_\eta(Y,X)$. Prove that for all $V \in X(M)$, the tensor $\nabla_V S_\eta$ is symmetric.

Hint: Differentiating $\langle S_\eta X, Y\rangle = \langle X, S_\eta Y\rangle$ with respect to V, we obtain

$$\langle \nabla_V(S_\eta X), Y\rangle + \langle S_\eta X, \nabla_V Y\rangle = \langle \nabla_V X, S_\eta Y\rangle + \langle X, \nabla_V(S_\eta Y)\rangle.$$

Using the fact that

$$\langle (\nabla_V S_\eta)X), Y\rangle = \langle \nabla_V(S_\eta X), Y\rangle - \langle S_\eta(\nabla_V X), Y\rangle$$

and the previous expression, we obtain easily that

$$\langle (\nabla_V S_\eta)S, Y\rangle = \langle X, (\nabla_V S_\eta)Y\rangle.$$

Solution. Differentiating $\langle S_\eta X, Y\rangle = \langle X, S_\eta Y\rangle$ with respect to V to obtain

$$\langle \nabla_V(S_\eta X), Y\rangle + \langle S_\eta X, \nabla_V Y\rangle = \langle \nabla_V(X), S_\eta Y\rangle + \langle X, \nabla_V(S_\eta Y)\rangle\,. \quad (1)$$

We know that

$$\langle (\nabla_V S_\eta)X, Y\rangle = \langle \nabla_V(S_\eta X), Y\rangle - \langle S_\eta(\nabla_V X), Y\rangle\,. \quad (2)$$

Combining (1) and (2) to obtain

$$\begin{aligned}
&\langle (\nabla_V S_\eta)X, Y\rangle \\
&= \langle \nabla_V(S_\eta X), Y\rangle - \langle S_\eta(\nabla_V X), Y\rangle && \text{by (2)} \\
&= \langle \nabla_V X, S_\eta Y\rangle + \langle X, S_\eta(\nabla_V Y)\rangle && \text{by (1)} \\
&\quad - \langle S_\eta X, \nabla_V Y\rangle - \langle S_\eta(\nabla_V X), Y\rangle \\
&= \langle X, \nabla_V(S_\eta Y)\rangle - \langle S_\eta X, \nabla_V Y\rangle && \begin{array}{c}\text{since } \langle \nabla_V X, S_\eta Y\rangle = \langle S_\eta(\nabla_V X), Y\rangle \\ \text{by the operator } S_\eta \text{is self-adjoint}\end{array} \\
&= \langle X, \nabla_V(S_\eta Y)\rangle - \langle X, S_\eta(\nabla_V Y)\rangle && \text{since } S_\eta \text{ is self-adjoint} \\
&= \langle X, (\nabla_V S_\eta Y)\rangle && \text{by (2).}
\end{aligned}$$

Thus

$$\langle (\nabla_V S_\eta)X, Y\rangle = \langle X, (\nabla_V S_\eta)Y\rangle\,.$$

Therefore for all $V \in \mathfrak{X}(M)$, the tensor $\nabla_V S_\eta$ is symmetric.

Exercise 123 (Exercise 6.11 from [Car92].). Let $f\colon \overline{M}^{n+1} \to \mathbf{R}$ be a differentiable function. Define the *Hessian*, Hess f of f at $p \in \overline{M}$ as the linear operator

Hess $f\colon T_p\overline{M} \to T_p\overline{M}$, (Hess $f)Y = \overline{\nabla}_Y$ grad f, $Y \in T_p\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersuperface in $\overline{M}$ defined by $M = \{p \in \overline{M}; f(p) = a\}$. Prove that:

(a) The Laplacian $\overline{\Delta} f$ is given by

$$\overline{\Delta} f = \text{trace Hess } f.$$

(b) If $X, Y \in X(\overline{M})$,then

$$\langle (\text{Hess} f)Y, X\rangle = \langle Y, (\text{Hess} f)X\rangle.$$

Conclude that Hess f is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}, p \in \overline{M}$, given by $(\text{Hess} f)(X, Y) = \langle (\text{Hess} f)X, Y\rangle, X, Y \in T_p\overline{M}$.

(c) The mean curvature H of $M \subset \overline{M}$ is given by

$$nH = -\text{div}(\frac{\text{grad} f}{\mid \text{grad} f \mid}).$$

Hint: Take an orthonormal frame $E_1, ..., E_n, E_{n+1} = \frac{\text{grad} f}{|\text{grad} f|} = \eta$ in a neightborhood of $p \in M$ in $\overline{M}$ and use the definition of divergence in Exercise 8, Chapter 3, to obtain

$$\begin{aligned} nH = \text{trace} S_\eta &= \sum_{i=1}^{n} \langle S_\eta(E_i), E_i\rangle \\ &= -\sum_{i=1}^{n} \langle \overline{\nabla}_{E_i}\eta, E_i\rangle - \langle \overline{\nabla}_\eta \eta, \eta\rangle = \sum_{i=1}^{n+1} \langle \overline{\nabla} E_i \eta, E_i\rangle \\ &= -\text{div}_{\overline{M}}\eta = -\text{div}(\frac{\text{grad} f}{\mid \text{grad} f \mid}). \end{aligned}$$

(d) Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from (c) that the mean curvature H of such a hypersuperface is given by

$$H = -\frac{1}{n}\mathrm{div}N,$$

where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Solution. (a) By defintion, for every $Y \in T_p\overline{M}$,

$$\begin{aligned}\overline{\Delta} f(Y) &= \mathrm{div}\,\mathrm{grad} f(Y)\\ &= \mathrm{trace}\overline{\nabla}_Y \mathrm{grad} f\\ &= \mathrm{trace}\,\mathrm{Hess} f(Y)\end{aligned}$$

since Y was arbitrary, so we have $\overline{\Delta} f = \mathrm{Trace}\,\mathrm{Hess} f$.

(b) Let $p \in \overline{M}$ and $E_1, E_2, \ldots, E_{n+1}$ be a geodesic fram at the point p. The, for $i, j \in \{1, 2, \ldots, n+1\}$,

$$\begin{aligned}\langle (\mathrm{Hess} f)E_i, E_j\rangle &= \left\langle \overline{\nabla}_{E_i} \mathrm{grad}_f, E_j \right\rangle\\ &= \left\langle \overline{\nabla}_{E_i} \sum_{k=1}^{n+1} (E_k(f))E_k, E_j \right\rangle\\ &= \left\langle \sum_{k=1}^{n+1} [(E_k(f))\overline{\nabla}_{E_i} E_k + E_i(E_k(f))E_k], E_j \right\rangle\\ &= \left\langle \sum_{k=1}^{n+1} E_i(E_k(f))E_k, E_j \right\rangle\\ &= E_i(E_j(f)).\end{aligned}$$

Thus

$$\langle (\mathrm{Hess} f)E_i, E_j\rangle = E_i(E_j(f)). \tag{1}$$

On the other hand,

$$\begin{aligned}
\langle E_i, (\mathrm{Hess} f)E_j\rangle &= \left\langle E_i, \overline{\nabla}_{E_j}\mathrm{grad} f\right\rangle \\
&= \left\langle E_i, \overline{\nabla}_{E_j}\sum_{k=1}^{n+1}(E_k(f))E_k\right\rangle \\
&= \left\langle E_i, \sum_{k=1}^{n+1}[E_k(f)\overline{\nabla}_{E_j}E_k + E_j(E_k(f))E_k]\right\rangle \\
&= E_j(E_i(f)).
\end{aligned}$$

Hence

$$\langle E_i, (\mathrm{Hess} f)E_j\rangle = E_j(E_i(f)). \tag{2}$$

But

$$\begin{aligned}
0 &= \overline{\nabla}_{E_i}E_j - \overline{\nabla}_{E_j}E_i \\
&= [E_i, E_j] \\
&= E_iE_j - E_jE_i,
\end{aligned}$$

so

$$E_i(E_jf) = E_f(E_if).$$

Therefore, combining with (1) and (2) to obtain

$$\langle (\mathrm{Hess} f)E_i, E_j\rangle = \langle E_i, (\mathrm{Hess} f)E_j\rangle .$$

Since this holds for all $i, j \in \{1, 2, \ldots, n+1\}$, we conclude that Hess$f$ is self-adjoint.

(c) Let $p \in M \subset \overline{M}$. let $E_1, E_2, \ldots, E_n, E_{n+1} = \frac{\mathrm{grad} f}{|\mathrm{grad} f|} = \eta$ be a an orthogonal frame about p. We have

$$\begin{aligned}
\left\langle \overline{\nabla}_\eta \eta, \eta\right\rangle &= \frac{1}{2}\eta\langle \eta, \eta\rangle \\
&= \frac{1}{2}\eta(1) \\
&= 0.
\end{aligned}$$

Thus by definition of divergent and of the mean curvature,

$$\begin{aligned} nH &= \text{Trace} S_\eta = \sum_{i=1}^{n} \langle S_\eta(E_i), E_i\rangle \\ &= -\sum_{i=1}^{n}(\overline{\nabla}_{E_i}\eta, \eta) - \langle \overline{\nabla}_\eta \eta, \eta\rangle = \sum_{i=1}^{n+1} \langle \overline{\nabla}_{E_i}\eta, E_i\rangle \\ &= -\text{div}_{\overline{M}}\eta \\ &= -\text{div}(\frac{\text{grad} f}{|\text{grad} f|}). \end{aligned}$$

(d) Let M be an embedded hyppersurface. Then, locally, $M = f^{-1}(r)$ for some regular value r. hence, for each point $p \in M$, $\text{grad} f(p) \neq 0$, so the expression derived in (c) is well-defined on all of M. Hence $H = -\frac{1}{n}\text{div}N$ where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

Exercise 124 (Exercise 6.12 *(Singularities of a Killing field)* from [Car92].). Let X be a Killing vector field on a Riemannian manifold M. Let $N = \{p \in M; X(p) = 0\}$. Prove that:

(a) If $p \in N$, and $V \subset M$ is a normal neighborhood of p, with $q \in N \cap V$, then the radial geodesic segment γ joining p to q is contained in N. Conclude that $\gamma \cap V \subset N$.

(b) If $p \in N$, there exists a neighborhood $V \subset M$ of p such that $V \cap N$ is a submanifold of M (this implies that every connected component of N is a submanifold of M).

Hint: Proceed by induction, using (a). If p is isolated, noth_ing has to be dome. In the contrary case, let $V \subset M$ be a nor-mal neighborhood of p such that there exists $q_1 \in V \cap N$ and consider the radial geodesic γ_1 joining p to q_1. If $V \cap N = \gamma_1$, by (a), the proof is complete. Otherwise, let $q_2 \in V \cap N - \{\gamma_1\}$ and let $\gamma_2$be the radial geodesic joining p to q_2. Let $Q \subset T_pM$ be the subspace generated by the vectors $\exp_p^{-1}(q_1)$ and $\exp_p^{-1}(q2)$ and let $N_2 = \exp_p(Q \cap \exp_p^{-1}(V))$. Show that for all $t \in \mathbf{R}$, the restriction of the differential $(dX_t)_p$ of the flow $X_t : M \to M$, to Q, is the identity; conclude now that $N_2 \subset V \cap N$. Proceed in this way until the dimension of T_pM is exhausted.

(c) The codimension, as a submanifold of M, of a con-nected component N_k of N is even. Assume the fol-lowing fact: if a sphere has a non-vanishing differen-tiable vector field on it then its dimension must be odd (for a proof, see Armstring, [Ar], p. 198).

Hint: Let $E_p = (T_pN_k)^\perp$ and let $V \subset M$ be a normal neigh-borhood of p. Set $N_k^\perp = \exp_p(E_p \cap \exp_p^{-1}(V))$. Since, for all t, $(dX_t)_p \colon E_p \to E_{p,}$, we have that X is tangent to $N_{\frac{1}{k}}$. On the other hand, $X \neq 0$ is tangent to the geodesic spheres of $N_K^\perp$with cinter$p$. From the theorem mentioned above, the dimension of such a phere os odd. Hence $\dim N_K^\perp = \dim E_p$ is even.

Solution. (a) Let φ_t denote the flow of X. If $p \in N$ and $V \subset M$ is a normal neighborhood of p, with $q \in N \cap V$, then let γ be the radial geodesic joining p to q. Since $q \in N$, $\varphi_t(q) = q$ for all t, where the flow is defined. Now, since φ_t is an isometry on V, φ_t maps geodesics to geodesics, specifically, $\varphi_t(\gamma)$ is a geodesic. Since φ_t fixes p and q, $\varphi_t(\gamma)$ is a geodesic passing through p and q in V; by uniqueness of geodesics, then $\varphi_t(\gamma) = \gamma$. Therefore, for any $q' \in \gamma \cap V$, $\varphi_t(q') = q'$, so $X(q') = 0$. Thus, $\gamma \cap V \subset N$.

(b) We shall prove this by induction. If p is an isolated point, then $\{p\}$ is a neighborhood of p and $N \cap \{p\} = \{p\}$ is a 0-submanifold of M. Otherwise, let V ve a normal neighborhood of p such that there exists $q_1 \in V \cap N$ not equal to p. Let q_1 be the radial geodesic from p to q_1. By part (a), $\gamma_1 \cap V \subset N$. if $V \cap N = \gamma_1 \cap V$, then we are done, since $\gamma_1 \cap V$ is a 1-submanifold of M.

If $V \cap N \neq \gamma_1 \cap V$, then let $q_2 \in V \cap N \backslash \{\gamma_1\}$. Let γ_2 be the radial geodesic joining p to q_2. By the same argument as in (a), $\gamma_2 \cap V \subset N$.

Now let $Q \subset T_pM$ be the subspace generated by $\exp_p^{-1}(q_1)$ and $\exp_p^{-1}(q_2)$, let $N_2 = \exp_p(Q \cap \exp_p^{-1}(V))$. Suppose $v \in N_2$. Then $v = a\exp_p^{-1}(q_1) + b\exp_p^{-1}(q_2)$ for some $a, b \in \mathbb{R}$. Hence

$$\begin{aligned}
(d\varphi_t)_p(v) &= (d\varphi_t)_p(a\exp_p^{-1}(q_1) + b\exp_p^{-1}(q_2)) \\
&= a(d\varphi_t)_p(\exp_p^{-1}(q_1)) + b(d\varphi_t)_p(\exp_p^{-1}(q_2)) \\
&= a\exp_p^{-1}(q_1) + b\exp_p^{-1}(q_2) \\
&= v.
\end{aligned}$$

Since q_1 and q_2 are fixed by φ_t. Therefore, we have $(d\varphi_t)_p$ is the identity on Q. Therefore, if $q \in N_2$,

$$\begin{aligned}
\varphi_t(q) &= \exp_p((d\varphi_t)_p(\exp_p^{-1}(q))) \\
&= \exp_p(\exp_p^{-1}(q_1)) \\
&= q,
\end{aligned}$$

so $N_2 \subset V \cap N$.

If $V \cap N = N_2$, then we are done, since N_2 is a 2-submanifold of M. Otherwise if $V \cap N \neq N_2$, we pick $q_3 \in (V \cap N) \backslash N_2$ and iterating the above procedure. At each stage, either this alogrithm terminates with N_i, an i-submanifold of M, or we proceed to the next stage. Since we have only n-dimentions to work with, we see that this procedure must terminate, and so $V \cap N$ is a submanifold of M.

(c) Let $p \in N_k \subset N$. Let $E_p = (T_p N_k)^{\perp}$ and let $V \subset M$ be a normal neighborhood of p. Let $N_k^{\perp} = \exp_p(E_p \cap \exp_p^{-1}(V))$. Now, suppose $v \in E_p$. Then $(d\varphi_t)_p(v) \in T_p N_k$ and so $\exp_p((d\varphi_t)_p v) \in N_k \cap V$, meaning that $\varphi_t(\exp_p((d\varphi_t)_p v)) = \exp_p((d\varphi_t)_p v) = \varphi_t(\exp_p(v))$. However, since $\exp_p v \notin N_k$, it can never flow to a point in N_k. Thus we conclude that $(d\varphi_t)_p \colon E_p \to E_p$, so X is tangent to $N_k^{\perp}$. On the other hand, since p is unique point on $N_k^{\perp}$, where X vanishes, we know that $X \neq 0$ is tangent to the geodesic spheres of $N_k^{\perp}$centered at p. Thus X is a non-vanishing vector field on the geodesic spheres of $N_k^{\perp}$, which are homeomorphic to plain odd spheres. Therefore, by the given fact, these geodesics spheres must be odd-dimensional, so $N_k^{\perp}$ must be even-dimensional. Since $\dim N_k^{\perp} = \operatorname{codim} N_k$, we conclude that N_k has even codimention in M.

Exercise 125 (Exercise 7.2 from [Car92].). Let $\widetilde{M}$ be a covering space of a Riemannian manifold M. Show that it is possible to give $\widetilde{M}$ a Riemannian structure such that the covering map $\pi \colon \widetilde{M} \to M$ is a local isometry (this matric is called the *covering metric*). Show that $\widetilde{M}$ is complete in the covering metric if and only if M is complere.

Solution. Let $\pi \colon \widetilde{M} \to M$ ve a covering map. Since $\pi(\widetilde{M}) = M$, so for every $\widetilde{m} \in \widetilde{M}$, $\pi(\widetilde{m})$ belongs to M. Assume that $\{(U_\alpha, x_\alpha)\}$ is a maximal differntiable structure on M. Whence $(U_{\alpha_m}, \tau_{\alpha_m})$ is a parametrization of M at m. Since the differentiable structure is maximal, so U_{α_m} concides with the distinguished neighborhood U_m of m such that $\pi^{-1}(U_m) = U_\alpha V_\alpha$ where the V_α's are pariwise disjoint open sets and $\pi|_{V_\alpha}$ is a homeomorphism of V_α onto U_m. The family $\{(V_\alpha, \pi^{-1}|_{V_\alpha} \circ x_\alpha)\}$ is a differentiable structure on $\widetilde{M}$. The inner product induced on $\widetilde{M}$ defined by

$$\langle x, y \rangle_q = \langle d\pi_q(x), d\pi_q(y) \rangle_{\pi(q)}$$

for $q \in \widetilde{M}$, $x, y \in T_q(\widetilde{M})$ which defined a Riemannian structure on $\widetilde{M}$. Thus $\widetilde{M}$ has a Riemannian structure. For the inner product above, π is a local isometry. Since π is a local isometry, so M is a complete metric space

if and only if $\widetilde{M}$ is a complete metric space. Using the Hoft and Rinow Theorem, $\widetilde{M}$ is complete if and only if M is complete.

EXERCISE 126 (Exercise 7.8 from [Car92].). Let M be a complete Riemannian manifold, $\overline{M}$ a connected Riemannian manifold, and $f\colon M \to \overline{M}$ a differentiable mapping that is locally an isometry. Assume that any two points of $\overline{M}$ can be joined by a unique geodesic of $\overline{M}$. Prove that f is injective and surjective (and, therefore, f is a global isometry).

SOLUTION. Let $p \neq q$ anf $\gamma\colon I \to M$ be a geodesic, $\gamma(0) = p$ and $\gamma(1) = q$. Cover $\gamma(I)$ by open sets U_α where $f|_{U_\alpha}\colon U_\alpha \to f(U_\alpha)$ is an isometry, by caompactness, there is a finite covering of U_i's or equivalently, there is a partition of I where $0 = t_0 < t_1 < \cdots < t_n = 1$,

$$\begin{aligned}\frac{D}{dt}\left(\frac{d(f\circ\gamma)}{dt}|_{[t_i,t_{i+1}]}\right) &= d_f\left(\frac{D}{dt}\left(\frac{d\gamma}{dt}|_{[t_i,t_{i+1}]}\right)\right)\\ &= df(0) = 0.\end{aligned}$$

Therefore $\frac{D}{dt}(\frac{d(f\circ\gamma)}{dt}) = 0$ on I implying that $f\circ\gamma$ is a geodesic in $\overline{M}$ joining $f(p)$ to $f(q)$. If $f(p) = f(q)$, then $f\circ\gamma$ would be a closed geodesic, contradicting the uniqueness assumption. Thus f is injective.

Uniqueness of geodesic joining any two points of $\widetilde{M}$ implies that $\overline{M}$ is complete, then $\exp_p\colon T_p\overline{M} \to \overline{M}$ is surjective for any $q \in \overline{M}$.

Let $p \in M$ is fixed. There is an open neighborhood $U \subset M$ containing p such that $f|_U \to f(U)$ is an isometry. We have $f(\exp_p(v)) = \exp_{f(p)}(df_p(v))$ for all $v \in T_pM$. Let $q \in N$ be arbitrary. There is $q \in T_{f(p)}\overline{M}$ such that $\exp_{f(p)}(w) = q$. Since df_p is an isometry of vector spaces, there is a $u \in T_pM$ such that $df_p(u) = w$. Hence $f(\exp_p(v)) = \exp_{f(p)}(df_p(v)) = q$. Thus f is surjective.

EXERCISE 127 (Exercise 7.9 from [Car92].). Consider the upper half-plane

$$\mathbf{R}^2_+ = \{(x, y) \in \mathbf{R}^2; y > 0\}$$

with the Riemannian metric given by

$$g_{11} = 1, \qquad g_{12} = 0, \qquad g_{22} = \frac{1}{y}.$$

Show that the length of the vertical segment

$$x = 0, \qquad \epsilon \leq y \leq 1, \qquad \text{with } \varepsilon > 0,$$

tends to 2 as $\epsilon \to 0$. Conclude from this that such a metric is not complete. (Observe, nevertheless, that when $y \to 0$ the length of vector, in this metric, becomes arbitrarily large.)

Solution. Parametrize the vertical sugement by

$$\begin{aligned} \gamma(t)\colon [t, 1] &\to \mathbb{R}^n_+ \\ t &\mapsto \gamma(t) = (0, t). \end{aligned}$$

For a given Riemannian metric,

$$ds^2 = dx^2 + \frac{1}{2} dy^2.$$

So the length of the vertical segment is

$$\begin{aligned} l &= \int_t^1 \frac{1}{\sqrt{t}}\, dt \\ &= 2\sqrt{t}|_t^1 \\ &= 2 - 2\sqrt{\epsilon}. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{\epsilon \to 0} l &= \lim_{\epsilon \to 0} (2 - 2\sqrt{\epsilon}) \\ &= 2. \end{aligned}$$

Since when $y \to 0$, the length of vectors in this metric becomes arbitrary large, so this metric is not complete.

Exercise 128 (Exercise 7.10 from [Car92].). Prove that the upper half-plane $\mathbf{R}^2_+$ with the Lobatchevski metric:

$$g_{11} = g_{22} = \frac{1}{y^{2'}}, \qquad g_{12} = 0$$

is complete.

SOLUTION. The metric given by

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

The mapping $T_a(x, y) = (x+a, y)$ is an isometry. This implies we can move any point to other point of the form $(0, y)$. Next, the mapping $D_\lambda(x, y) = (\lambda x, \lambda y)$ is an isometry for $\lambda > 0$. Putting these together shows that the hyperbolic plane is homogenous. Use T_{-x} to move (x, y) to (x', y'), move (x, y) to $(0, y)$, then use $D_{y'/y}$ to move $(0, y)$ to $(0, y')$, then use $T_{x'}$ to mocw $(0, y')$ to (x', y'). Thus, the upper half-plane $\mathbf{R}^2_+$with the Lobatchevski metric is homogeneous. By Exercise 12 below, it is complete.

EXERCISE 129 (Exercise 7.12 from [Car92].). A Riemannian manifold is said to be *homogeneous* if given $p, q \in M$ there exists an isometry of M takes p into q. Prove that any homegeneous manifold is complete.

SOLUTION. The homogeneity of the manifold implies that there is $\epsilon > 0$ such that $\exp_p$ is defined on $B(0, \epsilon) \subset T_p(M)$, $\forall p \in M$. Indeed, suppose that for a given $p \in M$, the exponential map $\exp_p$ is defined on $B(0, \epsilon) \subset T_p(M)$. Then given $q \in M$ and $f \in Iso(M, g)$ with $f(p) = q$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \exp_p \uparrow & & \uparrow \exp_p \\ B(0,\epsilon) \subset T_p(M) & \to & B(0,\epsilon) \subset T_p(M) \end{array}$$

commutes by uniqueness of geodesics and the fact that isometries preserve geodesic. Therefore, any geodesics defined on a closed interval $[a, b]$ can be extended to $(a - \epsilon, b - \epsilon)$. That is, (M, g) is geodesically complete. Using Hoft-Rinow theorem to obtain M is complete.

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